Models of Hyperbolic Geometry

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Contents

Introduction

Geometry, as we know it today, has its roots in ancient treatise *Elements*, in which Euclid built the entire geometry of that time on the solid foundation of the five postulates. While the first four postulates were intuitive and brief, the same could not be applied to the fifth one, also known as the *Parallel postulate*, which due to its complexity soon became the target of criticism. During the period of 2000 years, quite a few mathematicians considered that the place of the controversial postulate was not among the postulates at all, but that it belonged to the propositions instead. Consequently, they tried to prove it from other postulates, and even though their attempts were doomed to failure, the efforts were not in vain, since they ultimately led to the discovery of the non-Euclidean geometries. By replacing the Parallel postulate with its negation, we may derive two geometries substantially different from the Euclidean geometry: hyperbolic and elliptic geometry. We restrict our attention only to the former one, given that the elliptic geometry requires additional changes in the first four postulates. The thesis aims to provide a better understanding of the hyperbolic geometry and to demonstrate that it is equally consistent as the Euclidean geometry. For this purpose, some of the most common models of hyperbolic geometry are introduced and analysed, namely the pseudosphere, Beltrami-Klein disk, Poincaré disk, Poincaré half-plane and hemisphere. Throughout our study, we take a synthetic approach rather than analytic, meaning that the hyperbolic geometry is gradually built in an elegant way starting with the primitive notions and axioms, without the use of a coordinate system and calculus. The emphasis is on the geometric representation of the content, and most of the proofs and new concepts are supported by diagrams created in GeoGebra software.

In Chapter 1 the question of the Parallel postulate is placed in a historical context, and we deal with events that influenced the occurrence of the hyperbolic geometry. We start with Euclidean axiomatic system, and after addressing the issues surrounding the fifth postulate, several propositions regarding Euclidean parallels, relevant for the further development, are selected from *Elements* and proved. Over the years, there appeared many alternative versions of the fifth postulate. We list a few of these substitutes, and for the most frequent one, the Playfair's axiom, we prove the equivalence with the Parallel postulate. Then we look more closely at Saccheri's contribution and construct Saccheri's quadrilateral that will be beneficial in the analysis of divergent parallels. The chapter ends with the background story of the three founders of hyperbolic geometry: Gauss, Bolyai and Lobachevsky.

In Chapter 2 our main concern are hyperbolic parallel lines. First, some of the principal results of hyperbolic geometry are summarised without proofs. One of those seemingly peculiar theorems asserts that not all parallels have a mutual perpendicular. We classify the parallel lines according to this property on *divergent parallels* which admit the common perpendicular and *asymptotic parallels* which do not admit such perpendicular. Characteristics of both kinds are comprehensively explored. Afterwards, we introduce the notion of the defect and establish the relation between the sum of the angles of a triangle and its area.

Chapter 3 is devoted to models of hyperbolic geometry. With the help of the models that are constructed within Euclidean geometry, we are able to visualise all different results derived in the second chapter. On the example of the pseudosphere, the first exhibited model, it is shown that hyperbolic space is negatively curved. Therefore, it is not possible to fit the entire hyperbolic plane into the Euclidean plane, and some properties have to be distorted. In order to preserve certain aspects, others need to be violated. With this in mind, we consider the advantages and disadvantages of Beltrami-Klein disk, Poincaré disk and Poincaré half-plane. The last model, the hemisphere, served as a tool to establish the isomorphism between Beltrami-Klein and Poincaré models.

The final chapter contains a brief discussion of whether the space around us can be described using hyperbolic geometry. Even though this area is still insufficiently explored, and the answer to the question is far from being resolved, hyperbolic geometry did have a great impact in shaping the views of society about the physical world. We examine its practical application and the role in Einstein's theory of general relativity. However, considering the full depth of this complex matter, the subject is barely scratched.

1 Historical background

Geometry, being one of the oldest branches of mathematics, has a long history of development. Due to its constant progress, the foundations of geometry have also been challenged and reformulated, especially in the last couple of centuries. Perhaps the most precise and rigorous collection of statements on which the whole geometry is currently based is Hilbert's axiomatic system. David Hilbert (1862–1943) notably contributed to the establishment of the formal treatment of geometry. He actually modified and filled in the gaps of one much older system, which will be of greater interest for us and the backbone of our study.

1.1 The foundation of Euclidean geometry

Modern axiomatic geometry originated with Euclid, a Greek mathematician who lived and worked in Alexandria, around 300 BC. What made Euclid so significant is his book *Elements*, in which he combined all knowledge of geometry and number theory that had been developed up to his time. It certainly was not the first attempt to gather all known mathematics in a single volume, but it was the most successful and the only one from that period which has been preserved. The *Elements* consists of 13 books (i.e. chapters), and the importance lies in their deductive structure. Euclid started by defining basic concepts, such as point, line and angle. However, some of these terms are better left undefined, as in Hilbert's axiomatic system in which point, line, plane, incidence, congruence and betweenness are undefined terms called *primitive notions.* Definitions in *Elements* were followed by five postulates and five common notions, statements which are taken to be true without proof. Whereas the postulates are assumptions about geometric figures, common notions are more general assumptions, suitable to other sciences as well. Today we do not distinguish between those terms, and they are simply called *axioms.* Euclid organised further mathematical content in a strict logical sequence so that each and every result could have been proved from the axioms and the previously proven results. That made the *Elements* the first mathematical coherent axiomatic system.

The standard translation of Euclid's common notions and postulates, taken from [7], is as follows:

Common notion 1. *Things which equal the same thing also equal one another.*

Common notion 2. *If equals are added to equals, then the wholes are equal.*

Common notion 3. *If equals are subtracted from equals, then the remainders are equal.*

Common notion 4. *Things which coincide with one another equal one another.*

Common notion 5. *The whole is greater than the part.*

Let the following be postulated:

Postulate 1. *To draw a straight line from any point to any point.*

Postulate 2. *To produce a finite straight line continuously in a straight line.*

Postulate 3. *To describe a circle with any centre and distance.*

Postulate 4. *That all right angles equal one another.*

Postulate 5. *That, if a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles.*

Common notions do not require further discussion. On the other hand, it might be useful to briefly analyse the postulates.

Out of the first few proofs, it is clear that Euclid tacitly assumed in *Postulate 1* that there is one and only one straight line through any two given points. *Postulate 2* declares that any line segment can be extended arbitrarily in both directions, beyond any specified length, as much as one desires. The infinity of a line is another Euclid's assumption which is not explicitly stated. However, we cannot blame him for leaving out such an important feature, since ancient Greeks did not think of infinite entities as we do now. In *Postulate 3*, a circle with any given centre and radius is, again, unique and arbitrarily large. Although, at first sight, *Postulate 4* seems too obvious, it gives us important information about the uniformity of the plane. It indicates that the plane cannot be bent in such a way that right angle in one part of the plane is not equal to the right angle located somewhere else. In a sense, the right angle became a standard of measurement, and only because of this postulate, we are able to compare other angles to the right one and say whether they are obtuse or acute. Recall that the right angle is defined as an angle which is congruent to its supplementary angle.

Initially, *Postulate 5* might seem slightly tangled. The meaning is illustrated in *Figure 1.1.* Line t represents what today would be called *transversal*, a line that intersects at least two other lines in the same plane at distinct points. By intersecting lines a and b, two pairs of "interior angles on the same side" are formed, angles α and β, and angles γ and δ. In our case, a pair of angles that add up to less than two right angles is pair α and β. Postulate 5 implies that lines a and b will eventually, when sufficiently produced, meet on the right side of transversal t.

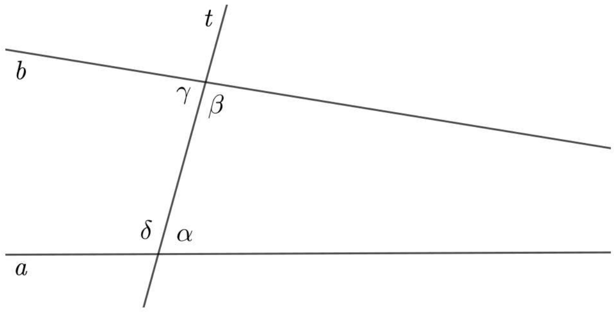

Figure 1.1: *Postulate 5*

It is important to emphasize that Postulate 5 does not say anything about the event when the sum of interior angles on the same side of transversal equals exactly two right angles. We may know empirically, from our prior experience, that lines a and b would be, in that case, parallel. But we need to be very careful here. Any mathematical statement that we make must be justified by what we already know to exist in our axiomatic system. Besides the laws of logic, we cannot refer to anything outside the axiomatic system or to something that has not been introduced yet. In order to stand behind our statement that lines a and b are parallel, we would have to be able to prove it by using axioms and statements that have already been proved. For now, we only have definitions and postulates, and that is not nearly enough to draw such a conclusion. In general, intuition and experience may be helpful in a way that we know what to expect to be true; but unless we can provide a legitimate proof for our hypothesis, we cannot claim that it is accurate. The best thing that we can do here is to apply the principle of *tabula rasa*, in other words, to start from scratch and rather forget everything we knew before. In fact, that is a precondition to accepting non-Euclidean geometry in the first place, taking into account all results that will oppose and challenge our common sense.
By and large, Postulate 5 only indicates when lines are *not* parallel. However, most of the properties regarding parallel lines are indeed consequences of Postulate 5, as we shall see later, and that is the reason behind calling it the *Parallel postulate*.

1.2 Under a cloud of suspicion

For an axiomatic system to be valid, a required condition is the consistency of the system. The system is consistent if no contradiction can be derived from the axioms, i.e. it is not possible to prove both a statement and its negation. Completeness and independence are two more properties of the axiomatic system. Even though they are not obligatory, it is better if they are satisfied. The system is said to be complete if any statement can be proved true or false, whereas independence means that no axiom can be deduced from the other axioms. While nobody argued that Euclid's axiomatic system is consistent and complete, independence, on the other hand, came into question.

The first four postulates are quite straightforward, brief and simple, as axioms should be. The fifth postulate, on the contrary, is not equally self-evident. It was never questioned if it is accurate, but whether it should be called an axiom. Although it appears as quite intuitive, it is considered to be a far more complex statement that lacks clarity. A significant number of mathematicians argued that Postulate 5 does not belong to postulates but rather to propositions. For more than 2000 years, they were trying to prove the controversial postulate using Euclid's definitions, remaining four postulates and propositions which do not depend on it. They did not succeed, but their attempts led to the discovery of non-Euclidean geometry. A list of mathematicians that have been actively engaged in looking for a proof of the fifth postulate is too long to cover all of them involved. Instead, we will later examine the achievements of only a few, whose contribution to the development of non-Euclidean geometry was the greatest. For the curious reader, [1] and [11] might be good sources of more complete historical overview.

It seems like Euclid himself had certain doubts regarding Postulate 5. He did his best to postpone using it as long as he could. The first proposition, the proof of which could not escape the fifth postulate, was *Proposition 29.* Although several propositions of the first twenty-eight ones could have been proved in fewer steps with the help of the fifth postulate, Euclid, for a reason best known to himself, took a longer path. Those first twenty-eight propositions along with *Proposition 31*, which do not require the fifth postulate to be proved, belong to *absolute geometry.* Absolute geometry, also referred to as neutral geometry, is based only on the first four of Euclid's postulates. Theorems of absolute geometry hold both in Euclidean and hyperbolic geometry, a form of non-Euclidean geometry.

1.3 Theory of parallels

Euclid, without a doubt, did a marvellous job in systemising all results that he had access to at that time, in a way that they naturally flow one from another. Beauty that is hidden within *Elements* comes to the surface in the following brief exposition of selected propositions. Proofs reflect elegance; they are simple and easy to follow. Nevertheless, some of the proofs have minor gaps because Euclid often relied on diagrams and unstated assumptions. For that reason Euclid is severely criticised today, and more precise proofs of the propositions are given, mostly based on Hilbert's axioms. Despite that, due to a desire to preserve a historical perspective on the development of geometry, rather than following Hilbert's rigorous axiomatic system, we will be consistent with Euclid's approach. The main results regarding parallels are stated, numbered and proved in the same manner as in the *Elements*. Only those propositions that are related to further content and can provide a better understanding of what is coming are selected. Since the numbering of definition and propositions corresponds with that of the *Book I* of *Elements*, they can easily be found in [7] and placed in its context.

Definition 23. *Parallel straight lines are straight lines which, being in the same plane and being produced indefinitely in both directions, do not meet one another in either direction.*

The definition itself, however, does not guarantee that parallel lines exist. As we shall see, the cornerstone in proving the existence will be *Proposition 16.* Outline of Euclid's proof of the following proposition will prove to be quite useful in the later analysis, and we will refer to it in *Section 1.5.*

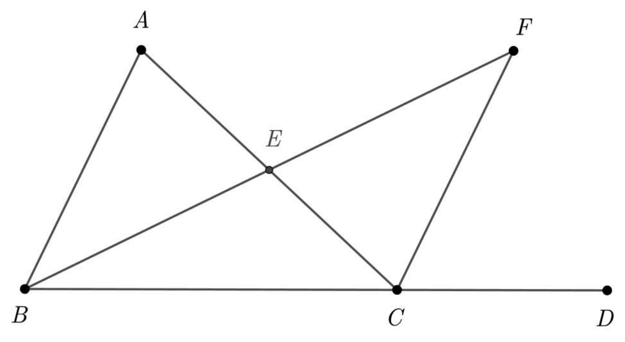

Figure 1.2: Proof of *Proposition 16.*

Proposition 16. *In any triangle, if one of the sides is produced, then the exterior angle is greater than either of the opposite interior angles.*

Proof. Let ABC be a triangle, and let side $\overline{BC}$ be extended to D (see *Figure 1.2*). We want to show that $\angle ACD > \angle CAB$ and $\angle ACD > \angle ABC$.
Let the point E bisect $\overline{AC}$. Join BE, and produce it to F so that $\overline{EF} = \overline{BE}$. Join FC.
Now we have

$$\begin{aligned} \overline{AE} &= \overline{EC} \\ \overline{BE} &= \overline{EF} \\ \angle BEA &= \angle FEC. \end{aligned}$$

By SAS (side-angle-side) congruence criterion, $\triangle ABE \cong \triangle CFE$. Hence $\angle CAB = \angle ACF$.
According to *Common notion 5*, the whole is greater than the part,

$$\angle ACD > \angle ACF,$$

and consequently,

$$\angle ACD > \angle CAB.$$

A similar argument shows that $\angle ACD > \angle ABC$, which proves the proposition. QED

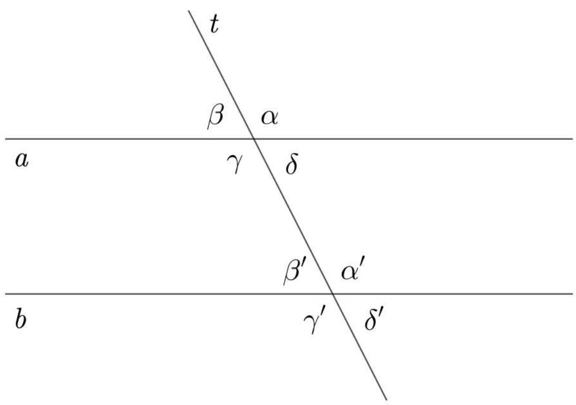

Figure 1.3: Angles formed by parallel lines cut by a transversal

Now we can use this result in proving the following proposition, in which parallel lines occur for the first time. Before we do that, it might be useful to explain Euclid's terminology regarding angles formed by a transversal. In *Figure 1.3* pairs of "alternate angles" are γ and α', δ and β'; pairs of "exterior angle" with corresponding "interior and opposite angle on the same side" are β and β', α and α', γ and γ', δ and δ'; and finally "interior angles on the same side": δ and α', γ and β'.

Proposition 27. *If a straight line falling on two straight lines makes the alternate angles equal to one another, then the straight lines are parallel to one another.*

Proof. Let t be a line that intersects two lines a and b at points A and B respectively, so that corresponding alternate angles α and β are equal (see *Figure 1.4*).
Suppose that a and b meet in point C, on the right side of transversal t. Then the exterior angle at vertex A of $\triangle ABC$ would be equal to the opposite interior angle at vertex B, which contradicts *Proposition 16*.
Analogously, one can show that a and b can not intersect on the left side of transversal t, either.
Therefore, it is impossible for a and b to meet, and they are, by definition, parallel. QED

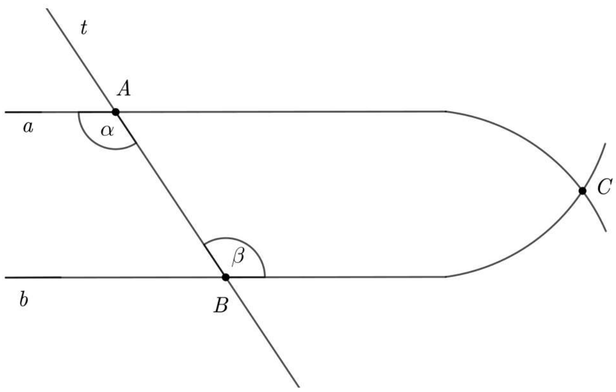

Figure 1.4: Proof of *Proposition 27.*

One might say that situation in *Figure 1.4* is impossible because it is not in the nature of straight lines to bend like that. Still, it is necessary to have in mind that point C could have been thousands of kilometres away from transversal t. Then on our limited area of the plane that we are able to see, lines a and b would appear to be parallel, but they would still have an intersecting point far in the distance. This is an example that justifies why illustrations are not acceptable means of proving mathematical statements.

Proposition 28 is similar to the previous one, and it states that lines a and b in *Figure 1.3* are parallel if exterior angle is equal to the interior and opposite angle on the same side ($\alpha = \alpha'$, $\beta = \beta'$, $\gamma = \gamma'$, $\delta = \delta'$) or if the sum of the interior angles on the same side equals two right angles ($\delta + \alpha' = \gamma + \beta' =$ two right angles).

At this moment, we know that parallel lines do exist, and we have everything we need to construct them. That is precisely what succeeding proposition is about.

Proposition 31. *It is possible to draw a straight line through a given point parallel to a given straight line.*

Proof. Let AB be a given line, and point P a given point not on AB.
Choose arbitrary point S on line AB, and draw line PS. Construct line CD through point P so that $\angle CPS$ is equal to $\angle PSB$ (see *Figure 1.5*).
By *Proposition 27*, lines AB and CD are parallel. QED

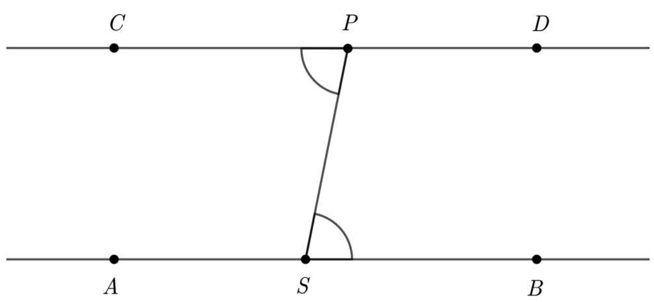

Figure 1.5: Proof of *Proposition 31.*

The order of propositions given by Euclid is not followed here; *Proposition 31* is placed before *Propositions 29* and *30*, and there is a reasonable explanation for doing so. Recall that all previous propositions hold in absolute geometry. On the other hand, to prove *Propositions 29, 30* and those after *Proposition 31*, invoking the fifth postulate is unavoidable. As they depend on the Parallel postulate, they are not valid outside Euclidean geometry.

Proposition 29 is a converse of *Propositions 27* and *28.* It asserts that if the lines are parallel and a transversal falls on them, then relations between angles will be the same as in premises of two indicated propositions.

Proposition 30. *Straight lines parallel to the same straight line are also parallel to one another.*

Proof. Let a, b, c be three distinct lines so that both a and b are parallel to c. Suppose, contrary to our claim, that a is not parallel to b. In that case they have to meet; denote the point of intersection by P. Choose random point S on c and join it with P. Since a and b are not the same line, in *Figure 1.6*

$$\alpha \neq \beta. \tag{1}$$

By *Proposition 29*, from $a \parallel c$ follows $\alpha = \gamma$. But also $\beta = \gamma$ because of $b \parallel c$. From this, we deduce that $\alpha = \beta$, which contradicts (1). Therefore, $a \parallel b$ and the proof is complete. QED

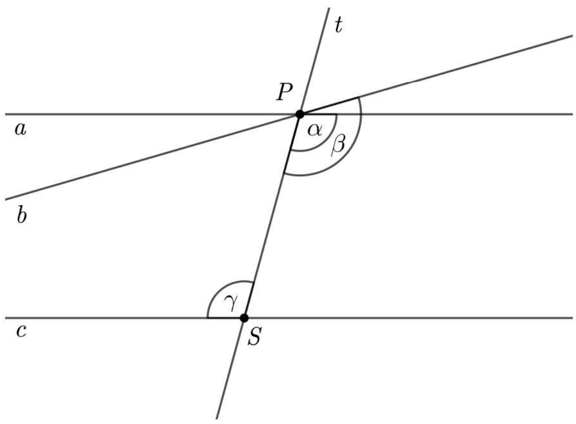

Figure 1.6: Proof of *Proposition 30.*

Euclid gave a proof that slightly differs from the one above. The reason behind taking a different road is that from this approach it might be easier to recognise the true meaning of the proposition. The direct consequence of *Proposition 30* is the uniqueness of parallels. Although that is not explicitly stated, the proposition ensures that there is one and only one parallel to a given line through a given point.

Like we have already said, *Proposition 30* does not belong to the absolute geometry, and therefore in absolute geometry we know that parallel lines exist, and we know how to construct them, but what we don't know is whether or not they are unique. This will be the fundamental question around which the whole discussion will be revolved.

1.4 Playfair's axiom

As mentioned earlier, the fifth postulate was not satisfying enough for the majority of mathematical society. In fact, quite a few mathematicians gave their own variations of the Parallel postulate. The most common alternative is known as Playfair's axiom, named after Scottish geometer John Playfair (1748–1819). Even though it was not his original finding, Playfair took credit for it, owing to the fact that axiom was widely adopted after it was included among five postulates in his 1795 edition of the *Elements*.

Playfair's axiom. *Given a line and a point not on the line, there exists exactly one line through the point parallel to the given line.*

It can be proved that Playfair's axiom is indeed equivalent to Postulate 5. Two statements in some axiomatic system are said to be equivalent if each one can be deduced from another, in the context of that axiomatic system. In other words, we need to determine a base that is mutual to both statements, assume one of the statements as an axiom, prove the other one as a theorem, and then reverse statements and repeat the steps. In our case, the base consists of Euclid's definitions, common notions, the first four postulates and all propositions that do not depend on the fifth postulate. Obviously, the base that we are looking for is absolute geometry.

Theorem 1.1. *Playfair's axiom is equivalent to Postulate 5.*

Proof. As discussed above, the proof is completed by showing that:

(I) Absolute geometry + Postulate 5 $\Rightarrow$ Playfair's axiom.

(II) Absolute geometry + Playfair's axiom $\Rightarrow$ Postulate 5.

Implication (I) follows from *Propositions 30* and *31* that were explained and proved in the previous section. What is left is to prove that (II) holds as well.

Let a and b be two lines such that, when they are cut by a transversal t at points S and P respectively,

$$\alpha + \beta < \text{two right angles,} \tag{2}$$

where α and β are interior angles on the right side of the transversal t, at points S and P respectively (see *Figure 1.7*). We want to prove that the lines a and b will

eventually meet on the right side of t.
Draw line c through P so that

$$\alpha + \gamma = \text{two right angles.} \tag{3}$$

Proposition 28 now implies $a \parallel c$. From (2) and (3) it can be concluded that $\beta < \gamma$, hence b and c are not the same line.
According to Playfair's axiom there can be only one line through P parallel with a, and consequently b has to intersect a. Let R be the point of intersection. If a and b met on the left side of t, then in the triangle PRS the sum of the interior angles at P and S would exceed two right angles because of (2). That is impossible by *Proposition 17* which states that in any triangle the sum of any two angles is less than two right angles. Therefore, lines a and b meet on the right side of transversal t as required. QED

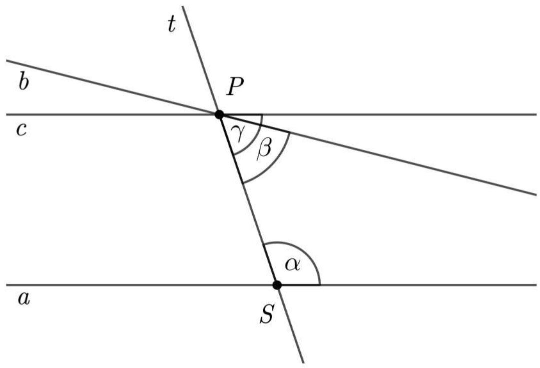

Figure 1.7: Proof of *Theorem 1.1*

So without changing the content of Euclidean geometry, Parallel postulate can be replaced by Playfair's axiom. That is exactly what happened in most of the books related to geometry, since Playfair's axiom is less technical, uses only the notion of parallelism and requires no information about the size of the angles. Simplicity and straightforwardness make Playfair's axiom considerably more appealing than the Parallel postulate. Another advantage that will prove highly useful is how easily it can be reformulated in order to be valid in non-Euclidean geometries as well.

Playfair's axiom is not the only statement logically equivalent to the fifth postulate in the presence of the axioms of absolute geometry. A surprisingly large number of the familiar theorems could take the place of the fifth postulate, and some of them are listed below.

Theorem 1.2. *The following statements are equivalent to the Parallel postulate:*

(a) There exist similar triangles which are not congruent.

(b) The sum of the angles of a triangle equals two right angles.

(c) Parallel lines are equidistant from one another.

(d) Any two parallel lines have a common perpendicular.

1.5 Saccheri

In the wide sea of unsuccessful endeavours to prove Postulate 5, one Italian priest left a significant trace. Girolamo Saccheri (1667–1733) had more creative ideas than the ones before him, and his strategy stood out among the rest. Saccheri's intention was to prove the Parallel postulate by *reductio ad absurdum*, assuming the contrary of the postulate and hoping to derive a contradiction. The centre of his work was quadrilateral[1] which is later called after him.

Definition 1.1. *A Saccheri quadrilateral is a quadrilateral in which a pair of equal opposite sides (arms) is perpendicular to one of the other sides which is called the base. The side opposite to the base is the summit, and the angles adjacent to the summit are the summit angles (see* Figure 1.8*).*

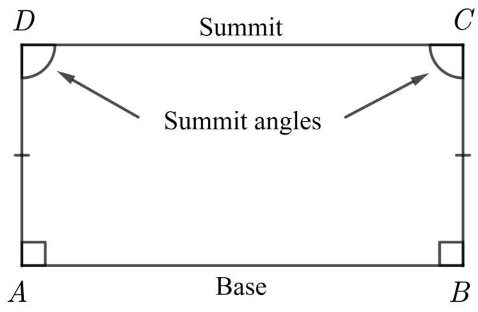

Figure 1.8: Saccheri quadrilateral

Theorem 1.3. *The summit angles of Saccheri quadrilateral are equal.*

[1]We shall deal with convex quadrilaterals only. Hence, the word "quadrilateral" would mean convex quadrilateral.

Proof. The proof is straightforward. Clearly, by SAS criterion $\triangle DAB \cong \triangle CBA$. This gives $\overline{AC} = \overline{BD}$, and so $\triangle ACD \cong \triangle BDC$ by SSS. QED

Saccheri proposed three possible hypotheses regarding summit angles:

1. Hypothesis of the acute angle.
2. Hypothesis of the right angle.
3. Hypothesis of the obtuse angle.

Afterwards, he wanted to avoid possible chaos in which, e.g. one Saccheri quadrilateral has acute summit angles, while another one has obtuse. The uniformity is established by our next theorem that is also called *Three musketeers theorem.* The proof is omitted for the sake of brevity, but the original version can be found in [10, Prop. V–VII].

Theorem 1.4 (Uniformity theorem)**.** *If one of the hypotheses stated above is true for a single Saccheri quadrilateral, it is true for every such quadrilateral.*

There is a close relationship between Saccheri quadrilateral and a triangle. For the proof of the following theorem see [6, p. 184].

Theorem 1.5. *The hypothesis of the acute (respectively, right, obtuse) angle is true if and only if the sum of the angles of a triangle is less than (respectively, equal to, greater than) two right angles.*

Saccheri was familiar with the equivalence of Postulate 5 with the statement that the sum of the angle of every triangle equals two right angles. Therefore, if he had managed to eliminate hypotheses of acute and obtuse angle, the Parallel postulate would have been proved. With absolute geometry as the base, he easily reached contradiction in the case of the obtuse angle.

Theorem 1.6 (Saccheri-Legendre Theorem)**.** *Assuming only absolute geometry, the sum of the angles of a triangle cannot be greater than two right angles.*

Proof. On the contrary, suppose that there exists $\triangle ABC$ in which the sum of the angles exceeds two right angles by a certain positive amount ε. We first apply Archimedean property which asserts that if a small positive quantity, in our case ε, is doubled often enough, it will eventually grow larger than any fixed amount. Let this fixed value be a measure of the angle $\angle CAB$. Thus Archimedean property assures that there exists $n \in \mathbb{N}$ such that $2^n\varepsilon > \angle CAB$. It follows that $\frac{1}{2^n}\angle CAB < \varepsilon$. The main idea of the proof is to find a triangle with the same angle sum as of the given triangle $\triangle ABC$, but with one of the angles less than ε, that is, less than or equal to $\frac{1}{2^n}\angle CAB$. Then the sum of the other two angles in $\triangle ABC$ would be greater than two right angles, which is a contradiction to *Proposition 17* that states that the sum of any two angles in any triangle is less than two right angles, and the proof would be complete.

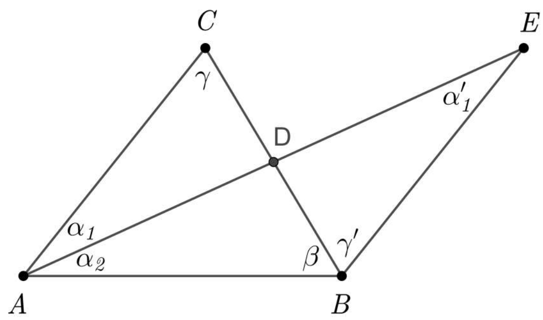

Figure 1.9: Proof of *Theorem 1.6*

Let D be the midpoint of $\overline{BC}$ in the given $\triangle ABC$. Extend AD to E so that $\overline{AD} = \overline{DE}$. $\triangle ADC \cong \triangle EDB$ by SAS, and consequently $\triangle ABC$ and $\triangle ABE$ have equal sum of the angles: $\alpha_1 + \alpha_2 + \beta + \gamma = \alpha_1' + \alpha_2 + \beta + \gamma'$ (see *Figure 1.9*).

Since $\alpha_1 + \alpha_2 = \angle CAB$, and $\alpha_1 = \alpha_1'$, at least one of the angles α_2 and α_1' does not exceed $\frac{1}{2}\angle CAB$. Without loss of generality, we can assume $\alpha_2 \leq \frac{1}{2}\angle CAB$. Then by finding the midpoint of $\overline{BE}$ in $\triangle ABE$, and by repeating all of the steps from earlier, we obtain a new triangle, with the same angle sum as of $\triangle ABC$ but with one of the angles less than or equal to $\frac{1}{2^2}\angle CAB$. If we continue in this manner, after n iterations, one of the angles of the newly obtained triangle would be less than or equal to $\frac{1}{2^n}\angle CAB$. Therefore, we have found a triangle with one of the angles less than ε but with the angle sum equal to that of $\triangle ABC$, which is the desired conclusion. QED

One part of the proof was inspired by Euclid's strategy and might seem familiar. It is worth recognising that the step in which a line segment was extended by its own length was applied earlier in proving *Proposition 16*. It is assumed here that we can always obtain a new line segment twice as long as the original one. But let us introduce the following model of geometry on the sphere (called the *spherical* *geometry*), illustrated in *Figure 1.10*, and see whether our assumption is valid.

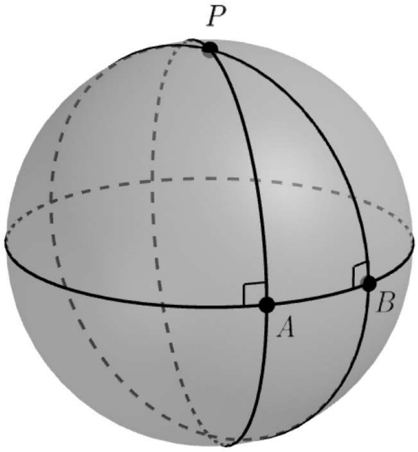

Figure 1.10: Spherical geometry

Consider geometry on the surface of a sphere in which "lines" are interpreted as great circles. A great circle is the intersection of the sphere and a plane that passes through the centre of the sphere. Two great circles intersect in two diametrically opposite points that are called antipodal points (e.g. the north and the south poles). An immediate issue arises now, as the first postulate is no longer satisfied. According to *Postulate 1*, there is one and only one line through any two points, and antipodal points lie on indefinitely many lines. To resolve this, we define a "point" to be a pair of two antipodal points. Identifying antipodal points can be imagined as pasting them together so that they merge into a single point. In this way, we obtain a new geometry called rather *elliptic geometry*.

Nevertheless, even with the modification stated above, troubles with the postulates are not brought to an end because the second postulate also poses a problem. Since a great circle does not have beginning nor end, a line in this model is unbounded. However, it has a finite length. A direct consequence is that not every line segment can be doubled without exceeding that certain length. In terms of the last proof, if segment $\overline{AD}$ was long enough, it would be possible that reflected point E lies on the initial segment $\overline{AD}$.

If that one particular step of both of the proofs cannot be carried out in the spherical geometry, one may suspect that *Proposition 16* and *Theorem 1.6* will not hold either. To confirm this, it is convenient to analyse a special case of triangles on the sphere. Let point P be the pole of a line AB. A point is called the pole of a line if every line through that point is perpendicular to the given line. Therefore, lines PA and PB are both perpendicular to AB (see *Figure 1.10*). Now, it is fairly easy to see that in the $\triangle PAB$ exterior angle at vertex A is equal to the interior and opposite angle at B. Likewise, when it comes to the sum of the angles of $\triangle PAB$, even without $\angle BPA$, angles $\angle PAB$ and $\angle ABP$ add up to two right angles.

Our example demonstrated rather strikingly that there exists geometry in which the sum of the angles of a triangle exceeds two right angles. However, Saccheri did nothing wrong by eliminating that possibility in the context of absolute geometry, since we have seen that not all postulates of absolute geometry hold on the sphere. What he did next, on the other hand, was a tremendous mistake and cost him the honour of being the one who discovered the non-Euclidean geometry. While he struggled to rule out the hypothesis of the acute angle, he derived many results, some of which will become initial theorems of hyperbolic geometry. Still, he refused to accept the idea that there exists a geometrical system in which the acute angle hypothesis is valid. Although he failed to reach any logical contradiction, he concluded: "The hypothesis of the acute angle is absolutely false; because [it is] repugnant to the nature of the straight line" [10, p. 155]. Nonetheless, there is no doubt that his work was of great significance. He inspired those who came after him, and eventually, it was verified that each of the three Saccheri's hypotheses leads to different geometry: hyperbolic, Euclidean and elliptic, respectively.

1.6 The discovery of non-Euclidean geometry

The sum of the angles of a triangle was analysed by yet another mathematician, Carl Friedrich Gauss (1777–1855), who was undisputedly the most notable mathematician of his time. Unlike Saccheri, he did not dismiss the possibility of the triangles with the angle sum less than two right angles. From his correspondence with other mathematicians, it is clear that he accepted the existence and validity of a geometry other than Euclidean. In one of his private letters written in 1824, he stated:

> The assumption that the sum of the three angles is less than 180° leads to a curious geometry, quite different from ours [the Euclidean], but thoroughly consistent, which I have developed to my entire satisfaction. ... The theorems of this geometry appear to be paradoxical and, to the uninitiated, absurd; but calm, steady reflection reveals that they contain nothing at all impossible. [6, pp. 243, 244]

Although Gauss had been working on non-Euclidean geometry since the age of 15, he did not publish anything on the matter. It is believed that the reason behind it was the fear that his good reputation could be ruined. He was already recognized as one of the greatest mathematicians, and he was probably afraid that he might lose public respect. Gauss knew that his findings were rather controversial and that society, perhaps, was not ready yet for a revelation of that kind.

He must have been quite surprised when he received a copy of a mathematical treatise from his old friend Farkas Bolyai. In the appendix of that book (the *Tentamen*, 1832), Farkas's son and Hungarian army officer, János Bolyai (1802–1860), published his discoveries regarding new geometry, which were more or less the same as those of Gauss. Following statement shows how enthusiastic János was about his discovery: "Out of nothing I have created a strange new universe" [3, p. 102]. However, his euphoria did not last long. Soon, they received a response from Gauss. In the introductory paragraph of the letter, Gauss wrote:

> I dare not praise such a work, ... to praise it would amount to praising myself; for the entire content of the work, the path which your son has taken, the results to which he is led, coincide almost exactly with my own meditations which have occupied my mind for from thirty to thirty-five years. [6, p. 241]

János was very disappointed by this response, and for a reason only known to himself, he never published anything else regarding non-Euclidean geometry again, even though he had notes with numerous results that were not included in the appendix of his father's book.

A few years earlier, in 1826, Nicolai Ivanovich Lobachevsky (1792–1856), a Russian professor of mathematics at the University of Kasan, delivered a lecture on a geometry in which it is possible to draw more than one parallel to the given line through a point not on the line. In 1829 he published an article on the same subject, but since it was in Russian, his work did not receive much attention. A decade later, Lobachevsky wrote a book in German, and it was eventually found by Gauss. Surely, Gauss was already familiar with the whole concept, but he was astonished by Lobachevsky's approach that was different than his. He was so impressed that he took special care to make Lobachevsky one of the members of Göttingen Scientific Society, which was the centre of German mathematics at that time.

Although both Lobachevsky and Bolyai had a connection with Gauss, and the idea of non-Euclidean geometry occurred to them almost simultaneously, all three of them came to their conclusions independently. We have not mentioned precisely their individual accomplishments, as most of their surprising results will be explained in the following chapter. Unfortunately, neither Bolyai nor Lobachevsky stumbled on acceptance regarding their discoveries during the lifetime. Apparently, Gauss was right when he assumed that the public was not ready for revolutionary new ideas.

Today, the form of the non-Euclidean geometry described by Gauss, Bolyai and Lobachevsky is called the hyperbolic geometry, also referred to as Lobachevskian geometry. The hyperbolic axiomatic system includes definitions, postulates and theorems of absolute geometry with the addition of the following axiom, which replaces Euclidean parallel postulate.

Hyperbolic Parallel Postulate. *Given a line and a point not on the line, there exists more than one line through the point parallel to the given line.*

Since it is possible to have infinitely many parallels to a line through the point not on the given line, a question about the third variation of Playfair's axiom naturally arises. Is there a geometry in which parallel lines do not exist? The answer is affirmative. In fact, we already gave a model of this case. It is easily seen that in *Figure 1.10* any two great circles will inevitably intersect. Recall that in *Section 1.3* the existence of parallels was derived from the first four postulates. More precisely, the existence was proved by invoking *Proposition 16*, and we had a chance to see that this proposition does not hold on the sphere. As stated before, a geometry that satisfies following axiom is called elliptic geometry.

Elliptic Parallel Postulate. *Given a line and a point not on the line, there exists no line through the point parallel to the given line.*

German mathematician Georg Friedrich Bernhard Riemann (1826–1866) was a student of Gauss and the first one who recognized that geometry on the sphere is a type of non-Euclidean geometry. That is why elliptic geometry is sometimes called Riemannian geometry, although this term is usually used in a much broader sense. His work was elaborated by another German mathematician, Felix Klein (1849–1925). He divided elliptic geometry on a single and double elliptic geometry, depending on whether two lines meet in a single point or in two points. However, single elliptic geometry is more commonly called just *elliptic geometry*, while double elliptic geometry is more often referred to as *spherical geometry*. The model introduced earlier in which antipodal points are identified obviously belongs to single elliptic geometry.

We have already indicated in the previous section that elliptic geometry, unlike hyperbolic and Euclidean, is not founded on the four postulates of absolute geometry. Another set of modified axioms is required for an axiomatic system of elliptic geometry to be consistent. However, an extensive discussion on elliptic geometry is beyond the scope of this paper, and for more details, we refer the reader to [5] and [6].

In the succeeding chapters, the subject of our attention will be hyperbolic geometry. While it was easy to accept the Elliptic parallel postulate due to the evident example on the sphere, some doubts regarding Hyperbolic postulate might still exist. Hopefully, suspicion will be eliminated after we explain the behaviour of the parallel lines in the hyperbolic plane and introduce a few models. Models will play a major role here, as they will help us to visualise the properties and provide clarity in understanding them.

2 Fundamental results of hyperbolic geometry

As previously stated, the four postulates of absolute geometry with the addition of the Hyperbolic parallel postulate create the basis for hyperbolic geometry. Note that, except theorems of absolute geometry, we already have a few theorems unique to hyperbolic geometry. The Hyperbolic parallel postulate is the negation of Playfair's axiom, which is a substitute for the fifth postulate. Therefore, we can conclude that the negation of any statement equivalent to the Euclidean parallel postulate will belong to hyperbolic geometry. This gives us the following four theorems, which are negations of the statements of *Theorem 1.2.*

Theorem 2.1. *If two triangles are similar, then they must be congruent.*

Theorem 2.2. *The sum of the angles of a triangle is less than two right angles.*

Theorem 2.3. *There exist parallel lines which are not equidistant from one another.*

Theorem 2.4. *There exist parallel lines which do not have common perpendicular.*

These results might seem quite strange at first glance. Perhaps the most striking one is *Theorem 2.1* which gives a new congruence criterion (angle-angle-angle), and it asserts that if corresponding angles of two triangles are equal, those two triangles will necessarily be congruent. Angles determine not just the shape of a triangle, but also its size. Similar triangles which are not congruent, do not exist in hyperbolic geometry.

Rectangles and squares are other figures whose existence is denied. Bearing in mind that each quadrilateral can be divided into two triangles, this is an immediate consequence of *Theorem 2.2.*

Theorem 2.5. *The sum of the angles of every quadrilateral is less than four right angles.*

Corollary 2.1. *The summit angles of a Saccheri quadrilateral are acute.*

Hyperbolic parallel postulate implies that there is more than one parallel line to the given line through the point not on the line, but it does not indicate specifically how many parallels there are. Moreover, *Theorem 2.3* claims that the parallels are not equidistant, so how do they behave then, relative to one another? To answer these questions, we first need to classify parallels according to their characteristics.

There are two kinds of parallel lines, those which admit a common perpendicular and those which do not admit such a perpendicular. Properties of these two types of parallels differ significantly, and we will consider them separately.

2.1 Parallels with a common perpendicular

Most of the results of this section were developed by Saccheri, even though he was not aware of his achievements. Taking that into account, it should not come as a surprise that the centre of our analysis will be Saccheri quadrilateral.

Theorem 2.6. *The line joining the midpoints of the base and the summit of a Saccheri quadrilateral is perpendicular to both.*

Proof. Let S be the midpoint of the base of a Saccheri quadrilateral $ABCD$, and P the midpoint of the summit (see *Figure 2.1*). By *Theorem 1.3* and SAS criterion, $\triangle APD \cong \triangle BPC$. This gives $\overline{AP} = \overline{BP}$, and by SSS, $\triangle APS \cong \triangle BPS$. It follows that $\angle ASP = \angle BSP$, and since they are supplementary angles, they are clearly right angles.

A similar consideration applies to supplementary angles $\angle DPS$ and $\angle CPS$. By *Postulate 4* and SAS, $\triangle ASD \cong \triangle BSC$. Now we have $\triangle DSP \cong \triangle CSP$ by SSS, which implies $\angle DPS = \angle CPS$, and the proof is complete. QED

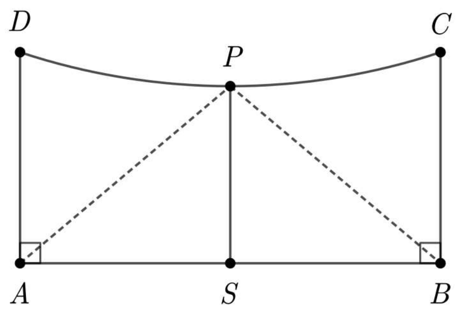

Figure 2.1: Proof of *Theorem 2.6*

As a consequence of *Proposition 27*, if two lines have a common perpendicular, they are parallel. Recall that *Proposition 27* is part of absolute geometry and thus holds in hyperbolic geometry. Therefore, the base and the summit of a Saccheri quadrilateral lie on parallel lines with a common perpendicular.

It is worth noticing that if two parallels have a common perpendicular, then they cannot have a second one. If they had, the quadrilateral formed by two parallel lines and their two common perpendiculars would be a rectangle, and that would contradict *Theorem 2.5.*

Corollary 2.2. *If two lines admit a common perpendicular, then that common perpendicular is unique.*

Some of the straight lines in diagrams in this chapter might seem curved, but that should not lead to a conclusion that straight lines really are curved in hyperbolic geometry. They are intentionally represented in that way to emphasise certain properties, for example, acute summit angles in Saccheri quadrilateral. The only purpose of diagrams is to help us to better understand the discussed content, and they should not be taken literally. Furthermore, if we consider the diagram in *Figure 2.1*, it is wrong to assume that it is possible to draw another straight line through points C and D beside this "curved" one that we have. In order to visualise some properties more accurately, we had to violate others. However, all results of absolute geometry are still valid. Straight lines remain straight, and through any two points still passes one and only one line.

Lemma 2.1. *Let $ABCD$ be any quadrilateral with right angles at A and B. Then $\angle BCD < \angle ADC \iff \overline{BC} > \overline{AD}$, that is, out of the two remaining angles, the greater angle is opposite the greater side.*

Proof. Suppose $\overline{BC} > \overline{AD}$. Then there is a point E on $\overline{BC}$ so that $\overline{BE} = \overline{AD}$ (*Figure 2.2*). By definition, $ABED$ is a Saccheri quadrilateral and $\angle ADE = \angle BED$ by *Theorem 1.3.* Since $\angle ADC$ is divided by line DE, $\angle ADE < \angle ADC$. $\angle BED$ being the exterior angle of triangle $\triangle ECD$ is greater than $\angle BCD$ (*Proposition 16*). Now we have $\angle BCD < \angle BED = \angle ADE < \angle ADC$.

Conversely, assume $\angle BCD < \angle ADC$. If sides $\overline{BC}$ and $\overline{AD}$ were equal, then $ABCD$ would be Saccheri quadrilateral, and angles at C and D would also be equal, which contradicts the assumption. This leaves us with two possibilities, either $\overline{BC} < \overline{AD}$ or $\overline{BC} > \overline{AD}$. If the first inequality was true, according to the first part of the proof, $\angle BCD$ would be greater than $\angle ADC$, and that is again contradiction with the premise. Thus $\overline{BC} > \overline{AD}$. QED

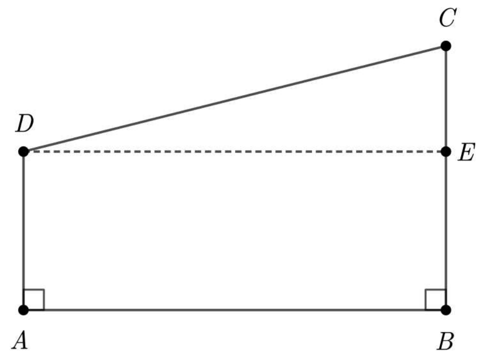

Figure 2.2: Proof of *Lemma 2.1*

Theorem 2.7. *In a Saccheri quadrilateral the summit is greater than the base, and the arms are greater than the segment joining the midpoints of the summit and the base.*

Proof. Let $ABCD$ be a Saccheri quadrilateral with right angles at A and B, and let S and P be the midpoints of the base and the summit, respectively (*Figure 2.3*). Since $\angle ADP$ is acute angle (*Corollary 2.1*) and $\angle DPS$ right angle (*Theorem 2.6*), previous lemma implies $\overline{AD} > \overline{SP}$. The same conclusion can be drawn for other arm $\overline{BC}$ and the second part of the theorem is proved.

Proof of the first part may be handled in much the same way. It is sufficient to make a slight change of the view and recognise that segments $\overline{AS}$ and $\overline{DP}$ are arms of the quadrilateral with the base $\overline{PS}$. Invoking previous lemma yields $\overline{DP} > \overline{AS}$. In the same manner, we can see that $\overline{PC} > \overline{SB}$. After combining these two inequalities, we have $\overline{DP} + \overline{PC} > \overline{AS} + \overline{SB}$, or $\overline{DC} > \overline{AB}$, and the proof is complete. QED

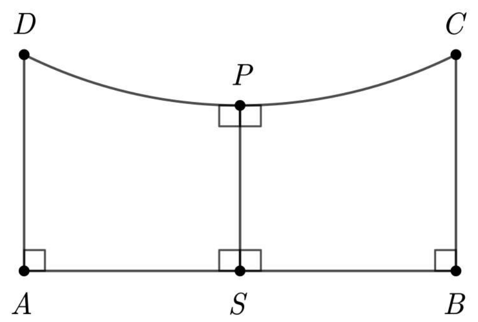

Figure 2.3: Proof of *Theorem 2.7*

In *Figure 2.3*, perpendicular SP divides the Saccheri quadrilateral into two congruent quadrilaterals $ASPD$ and $BSPC$. Such quadrilateral with three right angles is called the Lambert quadrilateral. It is named after Swiss mathematician Johann Heinrich Lambert (1728–1777), who tried to prove the fifth postulate by taking a similar path to the one Saccheri took. The remaining angle of Lambert quadrilateral is called the "fourth angle", and it can be acute, right or obtuse. Lambert was hoping to derive a contradiction in the case of acute and obtuse angle, which would force the Parallel postulate. Same as Saccheri, he only succeeded to eliminate the possibility of obtuse angle (which holds in elliptic geometry). In hyperbolic geometry, due to the *Theorem 2.5*, we can also rule out the hypothesis of the right angle.

Theorem 2.8. *The fourth angle in a Lambert quadrilateral is acute.*

Just like we can halve Saccheri quadrilateral to obtain Lambert quadrilateral, we can also carry out the reverse steps, and by reflecting Lambert quadrilateral over the arm adjacent to two right angles, we can produce Saccheri quadrilateral. Taking that observation into account together with *Theorem 2.7*, the following result is immediate.

Corollary 2.3. *In Lambert quadrilateral each side adjacent to the fourth angle is greater than the opposite side.*

From *Theorem 2.7*, it can be easily seen that parallel lines are not equidistant, as it is stated in *Theorem 2.3.* Recall that the distance from an external point to a line is defined as the length of the line segment that is perpendicular to the line and passes through the point. Lengths of segments $\overline{DA}$, $\overline{PS}$ and $\overline{CB}$ in *Figure 2.3* are three distances from the line DC to its parallel line AB. $\overline{DA}$ is equal to $\overline{CB}$ (arms of Saccheri quadrilateral), but they are greater than $\overline{PS}$ (*Theorem 2.7*). If parallel lines AB and DC were extended, the diagram suggests that the distance between the parallel lines would be the least when measured along the common perpendicular of those lines, and that the distance would increase symmetrically with respect to the common perpendicular. Therefore, the distance from one parallel line to another cannot have the same value in more than two points. Although we strictly emphasized earlier that conclusions should not be based on diagrams, in this case our intuition is on the right track, and these assumptions will, in fact, prove to be true.

Theorem 2.9. *The distance between two parallels with a common perpendicular is least when measured along that perpendicular. The distance from a point on either parallel to the other increases as the point recedes from the perpendicular in either direction.*

Proof. Let a and b be parallel lines with a common perpendicular intersecting a at point S and b at P (*Figure 2.4*). Choose arbitrary point B_1 on b (other than P), and let A_1 denote the foot of the perpendicular from B_1 to the line a. Since SA_1B_1P is a Lambert quadrilateral, $\angle A_1B_1P$ is acute and by *Lemma 2.1*, $\overline{A_1B_1}$ is greater than $\overline{SP}$, which proves the first part of the theorem.
Now choose another point B_2 on b such that B_1 is between P and B_2. A_2 denotes the foot of the perpendicular from B_2 to the line a. SA_2B_2P is a Lambert quadrilateral, and so $\angle A_2B_2P$ is acute. $\angle B_2B_1A_1$ is obtuse because of $\angle A_1B_1P$ being acute. Therefore, by *Lemma 2.1*, $\overline{A_2B_2} > \overline{A_1B_1}$.
A similar observation applies for distances from points on the line a to the line b. QED

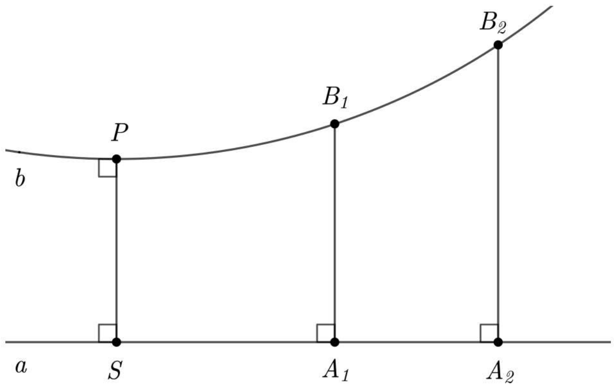

Figure 2.4: Proof of *Theorem 2.9*

The distance between parallel lines with a common perpendicular becomes infinitely large in either direction. Parallel lines diverge from each other on both sides of the common perpendicular, and that is the reason behind the terminology *divergently parallel lines*, which will be used from now on.

What remains is to determine how many lines divergently parallel to the given line is possible to draw through the point not on the line.

Theorem 2.10. *Given a line and a point not on the line, there are infinitely many lines through the point divergently parallel to the given line.*

Proof. Let a be the given line, P a point not on the line, and S the foot of the perpendicular from point P to the line a (see *Figure 2.5*). Through point P draw a line b perpendicular to the line SP. Then b is one divergently parallel line to the line a through the point P. On the line b choose arbitrary point B_1, and let A_1 be the foot of the perpendicular from B_1 to a. Previous theorem implies $\overline{A_1B_1} > \overline{SP}$. On $\overline{A_1B_1}$ there is a point P_1 such that $\overline{A_1P_1} = \overline{SP}$. SA_1P_1P is a Saccheri quadrilateral, and thus by *Theorem 2.6*, PP_1 is another divergently parallel line to a through P, different than PB_1. In the same manner, by choosing a different point B_2 on b, we can obtain a line PP_2 divergently parallel to the line a which differs from PP_1. To confirm that it is not the same line as PP_1, it is enough to notice that $\overline{SP} = \overline{A_1P_1} = \overline{A_2P_2}$, and we concluded earlier that at most two points at the time on a single line can be equidistant from its parallel line. Therefore, for each point on the line b, there exists a corresponding divergently parallel line to the given line a through P. QED

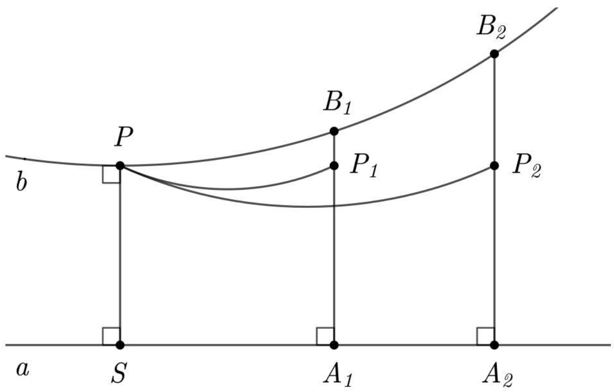

Figure 2.5: Proof of *Theorem 2.10*

2.2 Parallels without a common perpendicular

As we have seen, parallel lines in hyperbolic geometry are considerably different from those in Euclidean geometry. So far we have discussed only parallel lines which admit a common perpendicular. As we know, divergently parallel lines are not equidistant, and if we have the same setup as in the *Figure 2.5*, the angle between the line PS and a line divergently parallel to the line a through P, does not have to be the right angle (in fact, it will not be the right angle unless PS is the common perpendicular between those parallel lines, which is possible for only one divergently parallel line through P). It is reasonable to wonder whether there is a lower limit for this angle, for which after it is crossed, a parallel line becomes an intersecting line. How far can we rotate the parallel line through the point P towards PS in one direction in order that it is still parallel with the given line? In other words, is there a limiting line between divergently parallel and intersecting lines? The answer lies in another type of parallel lines, those without a common perpendicular, which are, for a good reason, also called *limiting parallel lines.*

Theorem 2.11. *If a is the given line, P a point not on the line, and S the foot of the perpendicular from P to a, then there exist exactly two distinct lines m and n through P, on opposite sides of PS, which do not meet a and have the property that every line through P lying within the angle between m and n that contains PS, intersects a, while every other line through P does not.*

To prove the existence of lines m and n described above, we will need Pasch's axiom of separation and Dedekind's axiom of continuity, which are two of the properties Euclid assumed without explicitly stating them.

Pasch's Axiom. *If a line intersects one of the sides of a triangle but does not pass through a vertex, it will intersect exactly one of the other two sides. If it does pass through one of its vertices, it will intersect the opposite side.*

Dedekind's Axiom. *For every partition of all the points on a line into two nonempty sets such that no point of either lies between two points of the other, there is a point of one set which lies between every other point of that set and every point of the other set.*

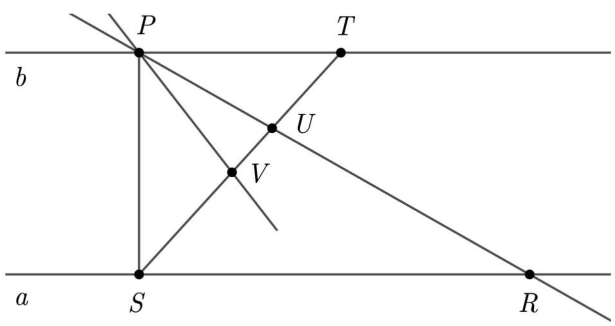

Figure 2.6: Proof of *Theorem 2.11*

Proof. Let a be the given line, P a point not on the line, and S the foot of the perpendicular from P to a. Construct a line b through P perpendicular to PS. Line b is divergently parallel to a. Let T be another point on b and join it with S. The points of $\overline{ST}$ can be partitioned into sets A and B such that each line joining P and a point of the set A intersects a, while each line joining P and a point of the set B does not intersect a.
Obviously, sets A and B are not empty, since S is in A and T is in B. Moreover, no point of one set lies between two points of the other, for if this was not true, we would have the following situation which is also described in *Figure 2.6*. Let U be the element of A (PU is an intersecting line) and V the element of B (PV is a parallel line). Let R be the intersection of PU and a. If V lies between U and S, then PV has to intersect SR by Pasch's axiom, which contradicts the assumption that V is the element of B.
Now we can invoke Dedekind's axiom. Therefore, on $\overline{ST}$ there is a unique point X, either in set A or in set B, such that PX divides parallels from intersecting lines.
To prove that X belongs to the set B assume the opposite, and denote the intersection of PX and a by Q. But then we can choose any point Z on a such that point Q is between S and Z, and in that case, PZ would intersect ST between X and T. This contradicts the fact that X is the limiting point between points of sets A and B. It follows that limiting line PX (let us denote it by n), does not meet the line a. By the same method, we can obtain the second limiting parallel line m on the other side of PS. QED

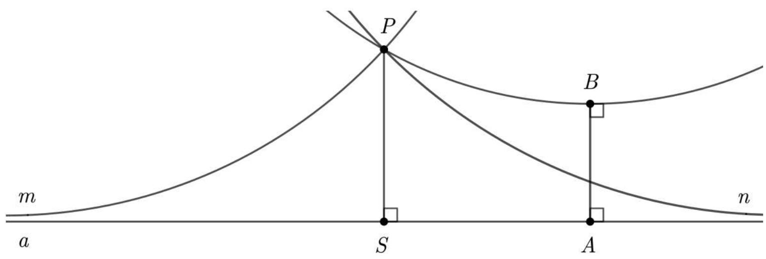

Figure 2.7: The parallels to a through P

Therefore, for a given line a and some external point P, there exist exactly two limiting parallels m and n, and infinitely many divergently parallel lines which lie between limiting parallels, within the angle that does not contain PS. This can be visualised as in *Figure 2.7*. It is convenient to distinguish between the two limiting parallels by calling m the *left*, and n the *right limiting parallel*, based on the side on which they emanate from point P towards line a.

Theorem 2.12. *With notation as in the previous theorem, the angles formed by PS and each limiting parallel line are equal and acute.*

Proof. Let the angle between PS and m be denoted by β, and the angle between PS and n by γ (see *Figure 2.8*). Suppose, contrary to our claim, that $\beta > \gamma$. Then there exists E in the interior of β such that $\angle SPE = \gamma$. By *Theorem 2.11*, PE meets a, say at F. On the line a on the other side of S construct point F' so that $\overline{FS} = \overline{F'S}$. By SAS criterion, $\triangle SPF \cong \triangle SPF'$. Thus $\angle SPF = \angle SPF' = \gamma$. This clearly forces that F' lies on n, and n intersects a, which is contradiction with n being the limiting parallel line. Hence, angles β and γ are equal.
To show that they are acute, we will again proceed by contradiction. Assume β and γ are obtuse. Construct line b through point P perpendicular to SP. By *Theorem 2.11*, the line b would intersect a, which is impossible since b is divergently parallel line. Also, the first part of the proof rules out the possibility that β and γ are right angles. Now when we know that β and γ are equal, they cannot be the right angles, as if they were, m and n would coincide, and we would have the Euclidean case with only one parallel line (divergently parallel lines would not exist due to *Theorem 2.11*). Therefore, angles β and γ are acute. QED

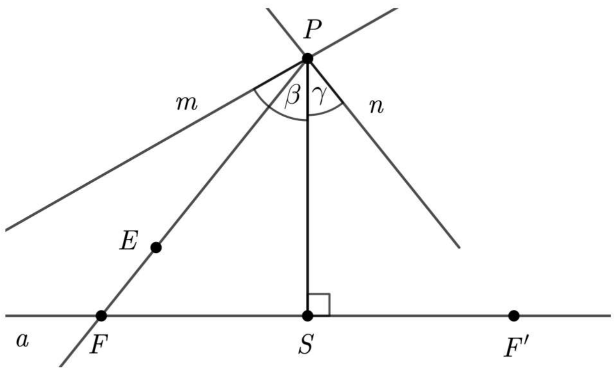

Figure 2.8: Proof of *Theorem 2.12*

Definition 2.1. *Given a line a and a point P not on a, let S be the foot of the perpendicular from P to a. The measure of the angle formed by the line SP and either of the two limiting parallel lines from P to a is called the* ***angle of parallelism****.*

The angle of parallelism is the foundation for hyperbolic trigonometry. While analysing it, Lobachevsky derived a number of trigonometric identities and formulae. The most interesting property is that the angle of parallelism depends only on the length of segment $\overline{SP}$. If the distance from any point to the given line is equal to the distance of any other point to the other line, the corresponding angles of parallelism are also equal. There are several nontrivial expressions that describe the connection between these two quantities. Furthermore, this relation between the length of a segment and the size of a corresponding angle of parallelism is inversely proportional: when the length of the segment increases, the angle of parallelism decreases. As the segment becomes arbitrarily great, the angle of parallelism approaches $0°$. And conversely, if the length of $\overline{SP}$ approaches 0, that is, if the given point P is in close proximity to the given line a, then the angle of parallelism approaches $90°$, and the limiting parallel line is very close to becoming a Euclidean parallel line.

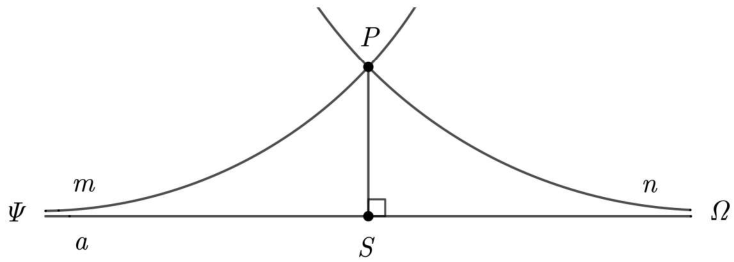

Figure 2.9: Limiting parallels asymptotically approach each other

Taking this into consideration, we can get some sense of how limiting parallel lines behave. While the distance between divergently parallel lines becomes arbitrarily great on both sides of the common perpendicular, limiting parallel lines diverge in one direction, but at the same time, they asymptotically approach each other in the opposite direction, meaning that they are gradually getting closer and closer but do not meet at any finite distance. This is why they are often called *asymptotic parallel lines.* We will prove this property in the following theorem, and it may be visualised as it is suggested in *Figure 2.9.* As the distance between the lines is approaching zero, we can think of them as having a point of intersection at infinity. Points at infinity are also called *ideal points.* All lines asymptotically parallel to each other are said to intersect at the same ideal point. For instance, if we observe some line and three of its right limiting parallel lines, all four of them are assumed to intersect at a single ideal point. Since every line has the right and left limiting parallel lines, every line will have exactly two ideal points, one in each direction, besides all the regular points it contains. In *Figure 2.9* ideal points of the line a are labelled with Ψ and Ω. However, it is important to emphasise that ideal points do not belong to the hyperbolic plane, and it is not correct to say that limiting parallel lines will eventually meet. It was simply convenient for mathematicians to introduce ideal points, and later this move turned out to be highly useful. The whole concept will be more clear when we consider it on the concrete models in the next chapter. Another surprising feature is that ideal points can also substitute vertices of a "triangle".

Definition 2.2. *A trilateral having one or more of its vertices at infinity is called an* ***asymptotic triangle****. Singly, doubly and triply asymptotic triangles have one, two and three vertices at infinity, respectively.*

In *Figure 2.9*, there are three asymptotic triangles, $SP\Psi$, $SP\Omega$ and $P\Omega\Psi$. These triangles have a lot of properties in common with ordinary triangles. Among them is the Pasch's axiom which will be needed in the proof of the following theorem.

Theorem 2.13. *Limiting parallel lines approach each other asymptotically in one direction and diverge in the other.*

Proof. Let n be the right limiting parallel line to the line a through the point P, and let S be the foot of the perpendicular from P to a. Denote by Ω the point of intersection at infinity of asymptotically parallel lines a and n. Choose any point R between S and P. Let m be the left limiting parallel to the line a through the point R. As m does not intersect a and cannot pass through the ideal vertex Ω of the asymptotic triangle $SP\Omega$ (it cannot be both left and right limiting paralllel to a), by Pasch axiom m will intersect n. Let Q denote the point of intersection, and let T be the foot of perpendicular from Q to a. Choose R' on the line n, so that Q is between P and R' and $\overline{RQ} = \overline{R'Q}$. Let S' be the foot of the perpendicular from R' to a. Lines m and n are limiting parallels to a through Q, so by *Theorem 2.12*, $\angle TQR = \angle TQR'$. By SAS criterion, $\triangle TQR \cong \triangle TQR'$, and $\overline{TR} = \overline{TR'}$. By *Common notion 3* and AAS criterion, $\triangle TRS \cong \triangle TR'S'$. Cosequently, $\overline{SR} = \overline{S'R'}$. Since $\overline{SR}$ can be arbitrarily small, we have proved that limiting parallel lines n and a asymptotically approach each other in one direction. To show that they diverge in the opposite direction, it is sufficient to choose point R on SP such that P is between S and R. The rest of the proof runs in the same way as before. The same conclusion can be drawn for the left limiting parallel to a through P. QED

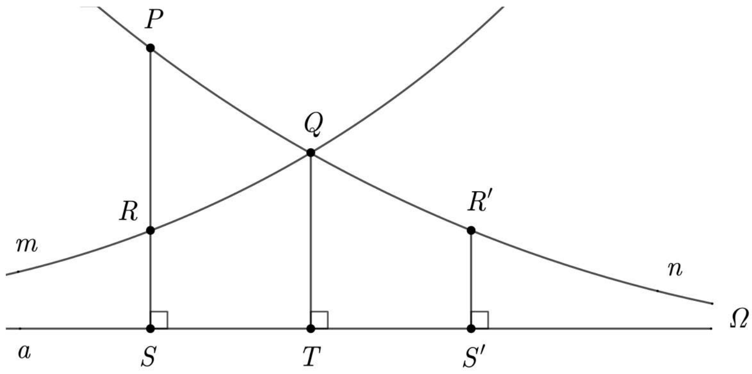

Figure 2.10: Proof of *Theorem 2.13*

2.3 The defect and area of a triangle

Before we head to the models, two more concepts need to be explained, and those are the defect and area of a triangle. If we recall that rectangles, and that includes squares, do not exist in hyperbolic geometry (*Theorem 2.5*), it is clear that the area cannot be measured in the same manner as in Euclidean geometry. Fortunately, there is a very elegant solution to this issue, and it involves the defect of a triangle.

As indicated earlier, the sum of the angles of a triangle in hyperbolic geometry is always less than two right angles (*Theorem 2.2*), and it is convenient to address the amount by which the angle sum differs from $2R$ [2].

Definition 2.3. *For any triangle $\triangle ABC$, the* **defect** *of $\triangle ABC$ is defined by*

$$\delta(\triangle ABC) = 2R - (\angle ABC + \angle BCA + \angle CAB).$$

By *Saccheri-Legendre Theorem*, the defect of every triangle is nonnegative. Clearly, in hyperbolic geometry it takes only positive values, while in Euclidean geometry it always equals zero, which is why the term is never being used.
Another straightforward property is that congruent triangles have the same defects, since corresponding angles of congruent triangles are equal. Furthermore, if a triangle is divided into two triangles, the defect of the original triangle is equal to the sum of the defects of the two triangles obtained by subdivision.

[2] In this section, R stands for the size of the right angle.

Theorem 2.14 (Additivity of the defect)**.** *For a triangle $\triangle ABC$, if D is any point between A and B, then*

$$\delta(\triangle ABC) = \delta(\triangle ADC) + \delta(\triangle DBC).$$

Proof. Let angles of triangles be denoted as in *Figure 2.11*.
By definition,

$$\delta(\triangle ADC) = 2R - (\alpha + \varphi_1 + \gamma_1) \quad \text{and} \quad \delta(\triangle DBC) = 2R - (\beta + \varphi_2 + \gamma_2).$$

It follows that

$$\begin{aligned} \delta(\triangle ADC) + \delta(\triangle DBC) &= 4R - (\alpha + \varphi_1 + \gamma_1 + \beta + \varphi_2 + \gamma_2) \\ &= 4R - (\alpha + \beta + \gamma_1 + \gamma_2) - (\varphi_1 + \varphi_2) \\ &= 2R - (\alpha + \beta + \gamma_1 + \gamma_2) \\ &= \delta(\triangle ABC). \end{aligned}$$

QED

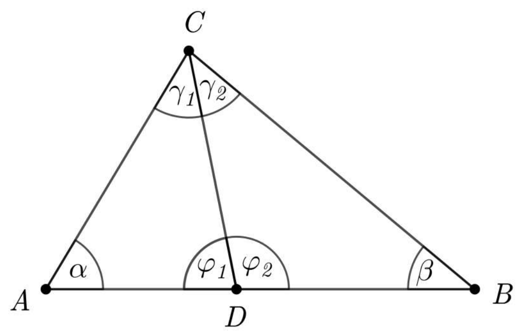

Figure 2.11: Proof of *Theorem 2.14*

It may be useful to remind ourselves of a few conditions that the area function must satisfy. It must be nonnegative, additive, and the areas of congruent triangles must be equal. Note that this coincides with the above introduced defect. Therefore, it makes sense to assume that the defect and area are closely related. Indeed, the area of any triangle, denoted by A, is proportional to its defect, with a proportionality constant depending on the unit of measurement.

Theorem 2.15. *There is a positive constant k such that for any $\triangle ABC$,*

$$A_{\triangle(ABC)} = k^2 \delta(\triangle ABC).$$

At first sight, it may seem rather unnecessary to work with the defect instead of calculating directly with the angle sum of a triangle. However, even though the defect is additive, the angle sum of a triangle is not, and by replacing the defect with the angle sum, the area of a triangle would not be well defined.

While in Euclidean geometry it is possible to construct a triangle of an arbitrarily large area, this will not be the case in hyperbolic geometry. Since $\delta(\triangle ABC) \leq 2R$ for every triangle, an immediate consequence of the previous theorem is the existence of an upper bound for the area of a triangle. This means that no matter how great the sides of a triangle might be, the area will always be less than this upper bound. The largest defect $2R$ (and thus the largest area) is obtained in a triply asymptotic triangle, whose all three vertices are at ideal points and hence all angles are zero. This can be nicely visualised in the models (see *Figure 3.21*).

Another peculiar feature that follows directly from *Theorem 2.15* is that the larger triangles have the greater defects, and their angle sum is close to zero, while the smaller triangles have the smaller defects, and the sum of their angles is approaching $2R$. Therefore, very small triangles (and very small parts of the hyperbolic plane) will behave almost like their Euclidean counterparts.

3 Models of hyperbolic geometry

In the previous chapter, we justified the existence and many properties of the two different kinds of parallel lines. However, even though theorems of hyperbolic geometry were logically deduced from the axioms, they still seem to go against our former experience. For someone whose eyes are accustomed to Euclidean geometry, it is extremely difficult to imagine most of the characteristics of parallels that were earlier explained. In spite of adjusting the diagrams so that they more accurately resemble properties that we needed, on certain occasions, parallel lines did not look parallel by any means, and it was self-evident that they would meet if they were sufficiently extended. Furthermore, some lines were represented as straight while others in the same diagram were curved, which served its purpose at that time but is not the best way to deal with the issue of visualising the concepts of hyperbolic geometry. In this chapter, we will overcome this obstacle by finally introducing the models of the hyperbolic axiomatic system. A model for a formal axiomatic system is a set of objects and relations that are defined in a specific way to correspond with primitive terms of the axiomatic system and to satisfy all of the axioms. As we build the rest of the system on those axioms, all further results that we obtained in theory will necessarily be true in the model. Moreover, a statement describing the model cannot be both true and false, and therefore if we are able to come up with a model for an axiomatic system, then that system is consistent.

Hence, by exhibiting the models of the hyperbolic axiomatic system, we will also confirm the consistency of the system. More precisely, our models will be constructed within Euclidean geometry, and in that way we will transfer the question of consistency from hyperbolic geometry to Euclidean. Primitive terms, such as point, line and plane, will be interpreted in Euclidean terms. Under this kind of interpretation, axioms of the hyperbolic system will be verified using the axioms and theorems of the Euclidean axiomatic system. Consequently, if any contradiction could be derived in hyperbolic geometry, then contradiction would inevitably be found in Euclidean geometry, as well. We can conclude that hyperbolic geometry is consistent only if the same applies to Euclidean geometry[3]. We will assume here that the Euclidean axiomatic system is indeed consistent. This has not been proved yet, and it is very unlikely that it will ever be proved. But that is the uncertainty

[3]The converse of the statement is also true. We can verify this by constructing a model of the Euclidean geometry within the hyperbolic geometry. This kind of model can be found in [6, p. 514].

that we can live with as there is no reason for suspecting otherwise.

Taking into account that Euclidean geometry has its limitations in comparison to hyperbolic geometry, it is understandable that by translating hyperbolic concepts into Euclidean, some properties have to be distorted. For this reason, there are different models that preserve different characteristics of the hyperbolic structure. We will discuss the most common ones. As we will see, some models may have their own properties that do not belong in the theory of hyperbolic geometry. If something exists in theory, then it has to be fulfilled in the model, but the converse of this statement is not always true. This is why it is not accurate to state that a certain model *is* a hyperbolic plane, it is only an interpretation of the plane. As we said earlier, model only assigns the meaning to the primitive terms of the axiomatic system in such a way that all of the axioms are true, and we have to be careful not to identify the model with the axiomatic system.

3.1 The pseudosphere

Around four decades after the discovery of hyperbolic geometry, in 1868, the first model was presented. This worthy accomplishment was achieved by Italian mathematician Eugenio Beltrami (1835–1900), whose only intention was to prove to the world, which chose to be oblivious towards new discoveries, that the geometry introduced by Lobachevsky, Bolyai and Gauss, was not a fiction. Beltrami recognised that a surface illustrated in *Figure 3.1*, which was already known to mathematicians at that time, can be placed in the context of hyperbolic geometry and serve as its model. The surface is called the pseudosphere, and it is obtained by rotating a *tractrix* around its asymptote.

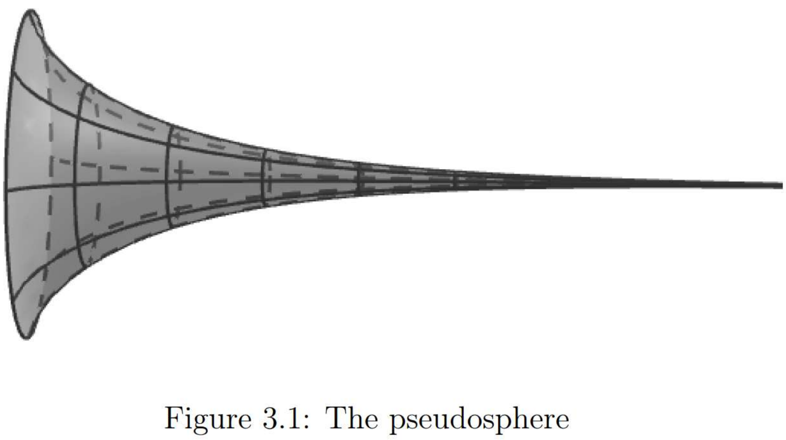

Figure 3.1: The pseudosphere

A tractrix is a very interesting curve also referred to as the "path of the obstinate dog". It will soon become clear why this name is so appropriate. Imagine a dog on a leash, and its owner, positioned as in *Figure 3.2*. Point S represents the owner, l is the road that he will follow, and a point P indicates the position of the dog. The leash PS has some fixed length, and the walk starts with PS perpendicular to l. As the owner walks along l, the stubborn dog resists, and it is being pulled by the owner, always behind him at the constant distance. The tractrix corresponds to the path of the dog.

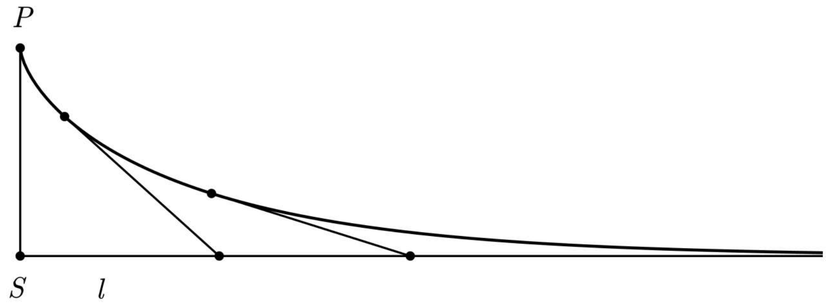

Figure 3.2: The tractrix

The pseudosphere is a surface of constant negative Gaussian curvature. We will try to explain very briefly what is meant by that, without getting too deep into the subject. To cover Gaussian curvature completely, a more detailed discussion is required, but it exceeds the scope of this paper.

In simple terms, Gaussian curvature indicates how a surface bends in the vicinity of a certain point. But before explaining the curvature of the surface, it is necessary to start with the curvature of the curve. Without any previous knowledge, we could still say with certainty that the straight line is not curved, while the circle is. Then, if we take a step further, we could say that a small circle is more curved than a larger one. It is intuitively clear that the sharper turning in the curve means larger curvature. Therefore, the curvature of the circle, denoted by k, can be described as the quantity inversely proportional to the radius of the circle, that is, $k = \frac{1}{r}$. The case with other curves is a little bit more complicated, as not all curves are equally curved at each of their points. Nevertheless, we can tackle this problem in a very simple way and use the above expression to calculate the curvature of any curve.

On a smooth curve choose an arbitrary point P. Then choose arbitrary points

P_1 and P_2 on the curve, one on each side of point P. There is a unique circle containing P_1, P and P_2. Fix the point P, and let P_1 and P_2 approach P. As we slide P_1 and P_2 towards the point P, we will obtain the circle that best approximates the curve in the vicinity of the point P, and shares with the curve the same tangent at P. This circle is called the osculating circle (see *Figure 3.3*). Now we can define the curvature of the curve at the point P as the curvature of the osculating circle. Note that if the curvature is not constant, the osculating circle will vary in size as it moves along the curve, and it may also shift from one side of the curve to the other. It is a matter of arbitrary convention which side of the curve determines positive curvature, and which side determines negative curvature. For instance, in *Figure 3.3* we could say that the curvature is positive if the osculating circle is below the curve, and negative if the osculating circle is above the curve.

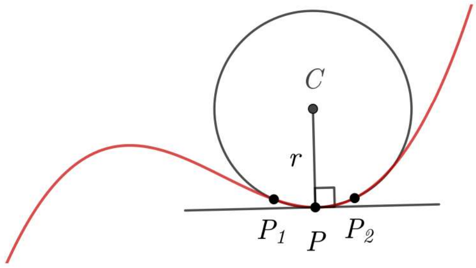

Figure 3.3: The osculating circle

Now we have the tools to determine the curvature of the surface. Choose an arbitrary point S on a surface. Let $\vec{v}$ be a tangent vector at S, and $\vec{n}$ a normal vector at S. A plane spanned by $\vec{n}$ and $\vec{v}$ intersects the surface in a curve called the normal curve of the surface at the point S in the direction $\vec{v}$. As $\vec{v}$ changes direction, a different normal curve is obtained. We are interested in two normal curves, one with the maximum curvature k_1, and the other one with the minimum curvature k_2. These two curvatures are called the *principal curvatures*, and they are achieved in perpendicular directions (see *Figure 3.4*). Finally, the Gaussian curvature of the surface at the point S, denoted by κ, is defined as the product of the principal curvatures: $\kappa = k_1 k_2$.

To determine instantly whether the surface has positive or negative Gaussian curvature at some point, we can simply place the tangent plane at the point and observe what is happening in the vicinity of that point. If the surface lies entirely on one side of the plane, then the surface has positive Gaussian curvature at the

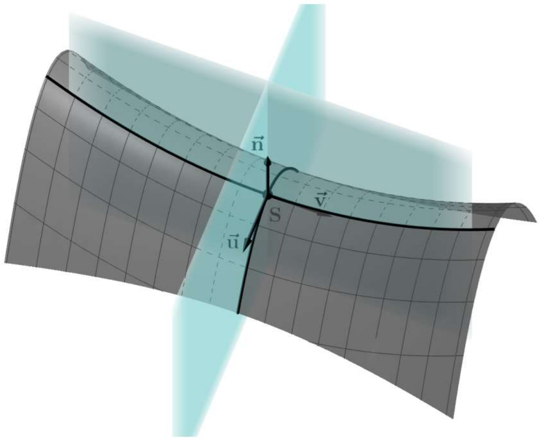

Figure 3.4: Principal curves of a saddle surface

point. On the contrary, if the tangent plane intersects the surface in the observed neighbourhood of the point, then the Gaussian curvature is negative.

A sphere is the surface of constant positive Gaussian curvature, a flat plane and cylinder have the curvature zero, while a saddle is the example of the surface with negative curvature. In general, positive Gaussian curvature is associated with elliptic geometry, zero curvature is associated with Euclidean geometry and negative curvature with hyperbolic geometry. This statement is not obvious and requires a proof, which is omitted due to its length and complexity.

As we have already stated, the pseudosphere has negative Gaussian curvature. The tractrix curves away from the axis of the pseudosphere while the circles of revolution curve towards it. The tractrix curves less and less as it approaches its asymptote (the axis), but at the same time circles of revolution get smaller and smaller. Consequently, as one principal curvature decreases, the other one increases. They perfectly balance each other, and moving sideways does not change anything. Therefore the product of principal curvatures (i.e. the Gaussian curvature) is constant. This is an essential property that makes a difference between a saddle surface (illustrated in *Figure 3.4*) and the pseudosphere. The curvature of the surface must be constant so that geometric objects can be moved from one part to another without changing the angles and the shape. The curvature of the saddle is negative, but not constant, and for that reason, it is not suitable for a model of hyperbolic geometry.

Now when we know the shape of the pseudosphere, we need to determine what are points, lines and plane in this model. A plane is interpreted as the surface itself, points are points on the surface, and lines are geodesics on the surface. A geodesic can be defined as a curve that contains the shortest path in the surface between two points. For instance, geodesics on the sphere are great circles. Pseudosphere, on the other hand, has three different types of geodesics: tractrix, circle, and rotating tractrix. The latter can be imagined as a curve wrapped around the entire pseudosphere any number of times.

One may argue that the lines do not look straight again, and considering how they are defined, it seems very difficult to confirm the Hyperbolic parallel postulate. Daina Taimina (born in 1954), a Latvian mathematician, shared this same attitude towards hyperbolic geometry. During her college, she struggled a lot with this subject. According to her, the hyperbolic geometry required too much imagination to make sense of it. After she passed the course, she hoped she will not have to deal with this abstract mathematical theory ever again. The irony occurred 20 years later when, as a professor at Cornell University, she was supposed to teach hyperbolic geometry. Not being left with many options, she had to find a way to physically experience the strange concepts she was not able to visualise. That was when she invoked her creative side and came up with the crochet model of the hyperbolic plane.

Figure 3.5: Crochet model of the pseudosphere

The model in *Figure 3.5* resembles a pseudosphere, with its wide part getting wavier as it spreads out. Hyperbolic lines can be easily sewed onto the crochet texture, and by folding the model, each of these lines can be made perfectly straight in Euclidean sense. Lines in *Figure 3.6* demonstrate that through one point there exist infinitely many lines parallel to a given line. They also manifest how two parallel lines are the closest at one point, and then how they diverge from each other on both sides. Taimina managed to show a lot of other properties of the hyperbolic plane with the help of the crochet method. How she did it and more about her work can be read in [12], the book from which the photos in *Figures 3.5* and *3.6* are taken.

Figure 3.6: Lines in the crochet model

If we had gone through all of the postulates in order to make sure that each one of them is satisfied, we would have disclosed that the pseudosphere actually does not provide a complete model of hyperbolic geometry. One problem lies in its boundary curve. In our description, the tractrix had a starting point, therefore the pseudosphere has a circular edge on one side, and geodesics cannot be extended infinitely in both directions. In the standard configuration, the pseudosphere is represented as not one, but two infinite horns, attached at the wide parts. Even though the pseudosphere would be infinite on both sides, not all geodesics could be continued smoothly over the edge where the two horns would meet. So that is not the way to get around the issue. Another problem is that the hyperbolic plane is simply connected, while the pseudosphere is not. Roughly speaking, a surface is simply connected if any closed curve on the surface can be continuously shrunk into a point. In the pseudosphere, this cannot be done with the geodesics around the axis of revolution. One possible solution to these two issues would be to wrap the pseudosphere around itself an infinite number of times and expand the surface infinitely beyond the rim. However, even this new surface, called the *universal cover* of the pseudosphere, coincides only locally with the hyperbolic plane, and hence it does not help either. In fact, according to David Hilbert, the problem regarding pseudosphere cannot be solved. He proved in 1901 that there exists no complete regular surface of constant negative curvature immersed in three-dimensional Euclidean space. Thus, the pseudosphere does not serve as a model of the entire hyperbolic plane, but only of the part of it. In this sense, the pseudosphere is comparable to the cylinder. In forming the cylinder, we take a line segment in Euclidean geometry, and at each endpoint of the segment, on the same side of it, we emanate perpendicular rays. Then we identify those two rays, that is, paste them together. The result is an infinitely long cylinder with a boundary, which represents only a part of the Euclidean plane. The procedure of constructing the pseudosphere in the hyperbolic plane is more or less analogous, and the pseudosphere can be thought of as being a hyperbolic cylinder.

Although it does not portray the ideal and flawless model, the pseudosphere has great historical importance. It is considered one of the milestones in the development of hyperbolic geometry. Due to it, there was huge progress in understanding the relations and concepts that exist exclusively in the hyperbolic plane, and it also initiated the construction of other models. We will proceed with the models that are all complete, but unlike the pseudosphere, not isometric (the distance in all of them is distorted). Nevertheless, they are more convenient for illustrating various hyperbolic objects and much simpler in comparison with the pseudosphere.

3.2 The Beltrami-Klein disk model

In the paper published in 1868, besides the pseudosphere, Beltrami also introduced a disk model of the hyperbolic axiomatic system. Since his work was based on differential geometry, the model more often carries the name of a German mathematician Felix Klein, who gave a more comprehensible description of the model in an article released in 1871.

To build the Beltrami-Klein disk model, we fix a circle in the Euclidean plane. The circle is called the *absolute circle*, and its interior represents the hyperbolic plane. Points in the model are Euclidean points within the absolute circle, not including the points on the boundary which are considered to be ideal points, or points at infinity. Points outside the absolute circle (“beyond infinity”) are called ultra-ideal points. Lines are interpreted as open chords of the absolute circle. The open chord can be defined as a line segment joining two points on the circumference of the circle, again not including the endpoints. Note that each line correlates with exactly two ideal points, just as we stated earlier.

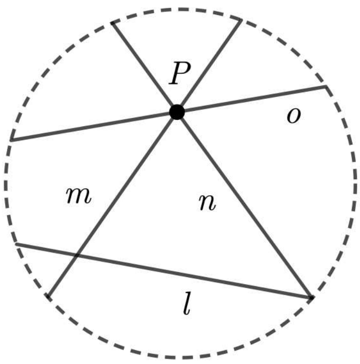

Figure 3.7: Lines in the Beltrami-Klein disk model

A few of the lines are illustrated in *Figure 3.7*, and we are interested in relations between line l and lines through the point P. Lines l and m clearly intersect at the hyperbolic point. On the contrary, lines l and n do not have an intersecting point in the hyperbolic plane, but the Euclidean chords on which they lie meet at an ideal point on the circumference of the absolute circle. Therefore we can think of l and n as meeting at infinity, which implies that they are asymptotically parallel. If lines l and o were extended, they would meet in an ultra-ideal point. However, since we restricted the hyperbolic plane on the interior of the absolute circle, and lines l and o do not intersect in an ordinary point and do not share an ideal point, they are divergently parallel. Obviously, through the point P, which is not on the line l, there are two asymptotically parallel and infinitely many divergently parallel lines to l.

Moreover, if the hyperbolic line in this model is perceived as a portion of a Euclidean line that lies inside the absolute circle, it is evident that for any two points in the circle there exists a unique hyperbolic line containing them. While the first postulate and the Hyperbolic parallel postulate are fairly straightforward, a certain amount of work is required to confirm the rest of them. Starting with the second one, we might wonder how can lines be infinite in both directions if they are bounded by the absolute circle. From our perspective, it seems impossible for lines to be longer than the diameter of that circle. The explanation lies in the way the distance between the points is defined in this model. For obvious reason, the Euclidean method of measuring length cannot be put into practice here, and it has to be carefully modified.

Before we give a new definition of distance, the concept behind it can be easily understood through the following story illustrated in *Figure 3.8*. Imagine a two-dimensional man living in the Beltrami-Klein disk and not being aware of the outside larger space. One day the man decides to take a long walk in one direction, and he brings a ruler with him. For the outside observer, the man would appear to be walking towards the boundary, and he would seem smaller and smaller as he moves away from the centre of the absolute circle. With each step, the man would shrink together with the ruler he carries with him. If the man occasionally measures his height, the height would be the same at the beginning of the walk and in the moment of measurement, and for him, nothing would be changed. The most important fact is that no matter how far the man walks, he will never be able to reach the boundary. For all we know, his path may be infinitely long, just as the lines in the model are infinite.

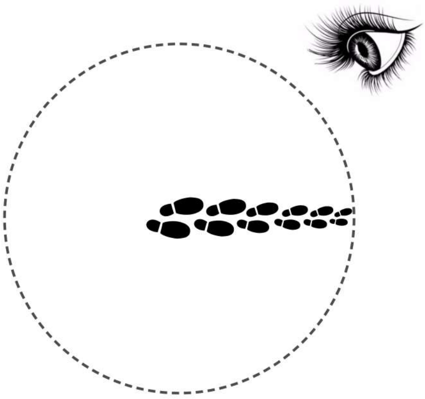

Figure 3.8: An infinite walk

Apart from the property that we have just described, the distance function, denoted by d, also needs to satisfy a couple of conditions (known as metric axioms). For any hyperbolic points A, B and C, we must have:

1. (Non-negativity) $d(A, B) \geq 0$ with equality if and only if $A = B$.
2. (Symmetry) $d(A, B) = d(B, A)$.
3. (Triangle inequality) $d(A, B) \leq d(A, C) + d(C, B)$.

In order to achieve an appropriate formula for the length that fulfils all of the above, we first need to introduce an auxiliary tool, called the *cross-ratio*.

Definition 3.1. *Given four collinear points A, B, C and D, we define the* ***cross-ratio*** (AB, CD) *by*

$$(AB, CD) = \frac{(AC)(BD)}{(AD)(BC)}$$

where (AC) stands for the Euclidean distance between points A and C.

Definition 3.2. *Let A and B be points inside the absolute circle, and let Ω and Ψ be the ideal points associated with hyperbolic line AB. Then the length $d(A, B)$ in the Beltrami-Klein disk model is defined by*

$$d(A, B) = \frac{1}{2} \left| \ln(AB, \Omega\Psi) \right|. \tag{4}$$

We will see that the definition is independent of the order in which A and B, or Ω and Ψ are listed. To show that with this metric the lines are indeed infinite, we just need to fix one of the points A and B, and let the other one slide towards Ω or Ψ. The cross-ratio $(AB, \Omega\Psi)$ will approach either zero or infinity, and in both cases, the distance between A and B will approach infinity.

Theorem 3.1. *The distance given by the formula (4) is well defined, that is, it satisfies metric axioms.*

Proof. The proof falls naturally into three parts.
Non-negativity. It is trivial that $d \geq 0$. The equality part is as easy to confirm:

$$d(A, B) = 0 \iff \ln(AB, \Omega\Psi) = 0 \iff (AB, \Omega\Psi) = 1 \iff A = B.$$

Symmetry. If $(AB, \Omega\Psi) = \lambda$, then by definition of the cross-ratio, $(BA, \Omega\Psi) = \frac{1}{\lambda}$. Now we have

$$\begin{aligned} d(B, A) &= \frac{1}{2} \left| \ln\left(BA, \Omega\Psi\right) \right| = \frac{1}{2} \left| \ln \frac{1}{\lambda} \right| = \frac{1}{2} \left| \ln \lambda^{-1} \right| = \frac{1}{2} \left| -\ln(\lambda) \right| = \frac{1}{2} \left| \ln(AB, \Omega\Psi) \right| \\ &= d(A, B). \end{aligned}$$

This means that the order of A and B in the cross-ratio in (4) does not affect $d(A, B)$. Since $(AB, \Omega\Psi) = (BA, \Psi\Omega)$, it does not matter in which order we write Ω and Ψ either.
Triangle inequality. A certain knowledge of projective geometry is required to prove the triangle inequality. This is why we will show only equality when A, B and C are collinear. For complete proof, we refer the reader to [8, p. 157].

For convenience, let A lie between B and Ψ, and let C lie between A and B (as in *Figure 3.9*). Due to this arrangement $(A\Omega) > (C\Omega)$, $(C\Psi) > (A\Psi)$, etc. Now the absolute value signs can be dropped because cross-ratios $(AB, \Omega\Psi)$, $(AC, \Omega\Psi)$ and $(CB, \Omega\Psi)$ are all greater than 1, and natural logarithm of each of them is positive. This gives

$$\begin{aligned} d(A,C) + d(C,B) &= \frac{1}{2}\ln(AC,\Omega\Psi) + \frac{1}{2}\ln(CB,\Omega\Psi) \\ &= \frac{1}{2}\ln\left(\frac{(A\Omega)(C\Psi)}{(A\Psi)(C\Omega)} \cdot \frac{(C\Omega)(B\Psi)}{(C\Psi)(B\Omega)}\right) \\ &= \frac{1}{2}\ln(AB,\Omega\Psi) \\ &= d(A,B). \end{aligned}$$

QED

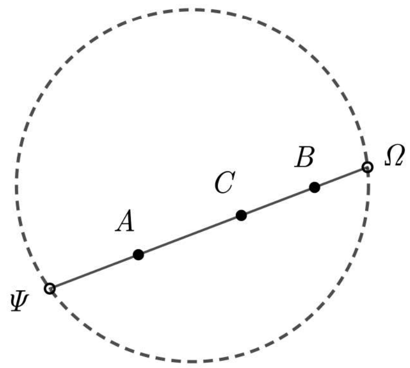

Figure 3.9: Proof of *Theorem 3.1*

The Beltrami-Klein disk model preserves the "straightness" of Euclidean lines, but not without the cost. Not only distance differs from its Euclidean counterpart, but so does angle measure. If we consider the asymptotic triangle in *Figure 3.7*, formed by lines l, m and n, the sum of its angles equals two right angles in the Euclidean sense, and we know this is not the case in hyperbolic geometry. In addition, the angle enclosed by two asymptotically parallel lines is supposed to be zero. This is one of the examples why we cannot measure angles in the way that we used to.

Instead of developing the whole new method of measurement, we will show at the end of this chapter that there exists a transformation between the Beltrami-Klein disk and the Poincaré disk model, in which the Euclidean angles are preserved. The transformation will map lines to lines, and it will be conformal, which means that the angle between two lines in the Beltrami-Klein model will correspond to the angle between their images in the Poincaré model.

At this moment, we will describe only the right angles and demonstrate how to construct a line perpendicular to a given line. There are two different cases to consider, depending on whether a line passes through the centre of the absolute circle. Both situations are illustrated in *Figure 3.10.*

Definition 3.3. *Let l and m be two lines in the Beltrami-Klein disk model and let at least one of them pass through the centre of the absolute circle. The lines l and m are said to be perpendicular if they are perpendicular in the Euclidean sense.*

Definition 3.4. *Let l be a line in the Beltrami-Klein disk model that does not pass through the centre of the absolute circle, and let t_1 and t_2 be the Euclidean tangents to the absolute circle at the ideal points associated with the line l. The Euclidean point of intersection of tangents t_1 and t_2 is called the* **pole** *of the line l, and we denote it by $P(l)$.*

Definition 3.5. *Let l and m be two lines in the Beltrami-Klein disk model such that neither of them passes through the centre of the absolute circle. The line m is said to be perpendicular to l if the Euclidean line extending m passes through the pole $P(l)$.*

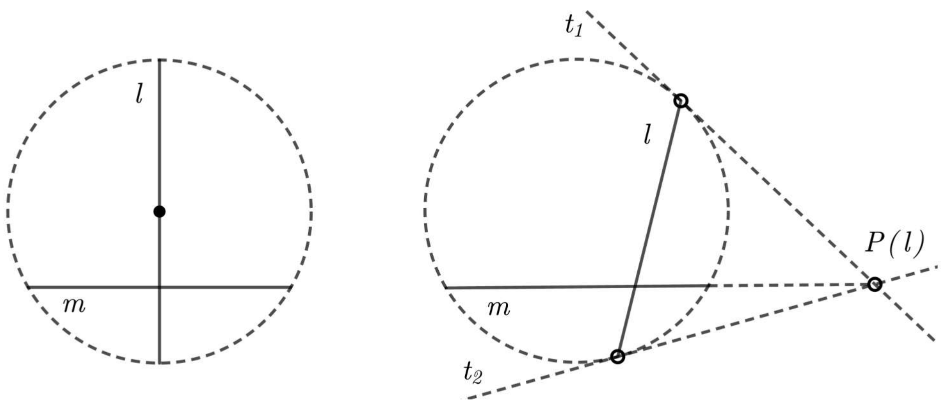

Figure 3.10: Perpendicular lines in the Beltrami-Klein disk model

Hence all lines perpendicular to l when extended pass through the pole $P(l)$. It is evident that the pole is ultra-ideal point, and that $l \neq m$ implies $P(l) \neq P(m)$.

Furthermore, the perpendicularity defined above is symmetric. If m is perpendicular to l, then l is perpendicular to m, and when extended, each of them passes through the pole of the other one.

There is one aspect of hyperbolic geometry that is perfectly demonstrated in this model. Let us consider lines that are divergently or asymptotically parallel, and observe what happens with their common perpendicular (if they have one). By *Definition 3.5*, the common perpendicular of two lines l and m should pass through both of their poles $P(l)$ and $P(m)$. It can be proved that if l and m are divergently parallel, a Euclidean line joining their poles intersects the absolute circle in two points. A part of that line that lies within the absolute circle gives the common perpendicular between l and m [4]. In fact, the ultra-ideal point at which divergently parallel lines meet if they are extended is the pole of their common perpendicular (see *Figure 3.11*).

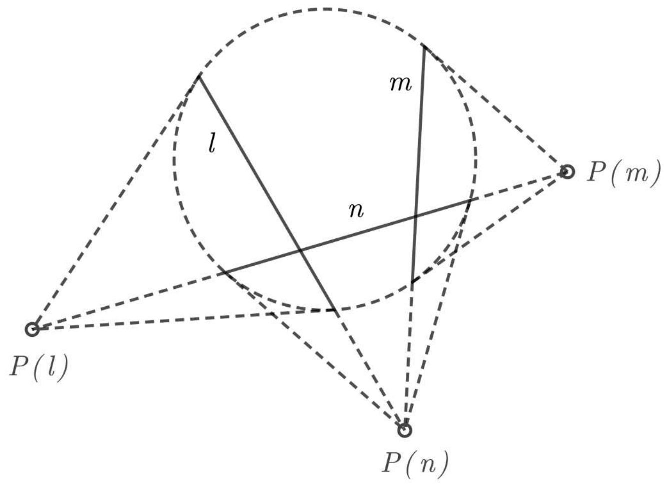

Figure 3.11: Common perpendicular between divergent parallels

However, if l and m are asymptotically parallel, the Euclidean line joining $P(l)$ and $P(m)$ is the tangent to the absolute circle at the ideal point in which l and m "meet". As no part of the tangent lies inside the absolute circle, asymptotic parallels do not have a common perpendicular (see *Figure 3.12*). This illustrates the theory from the previous chapter.

The only remaining segment in this model that should be clarified is the existence of circles. If we return to that two-dimensional being living inside the disk

[4] If one of the divergently parallel lines l and m is the diameter of the absolute circle, say l, then the common perpendicular passes through $P(m)$ and is perpendicular to l in the Euclidean sense.

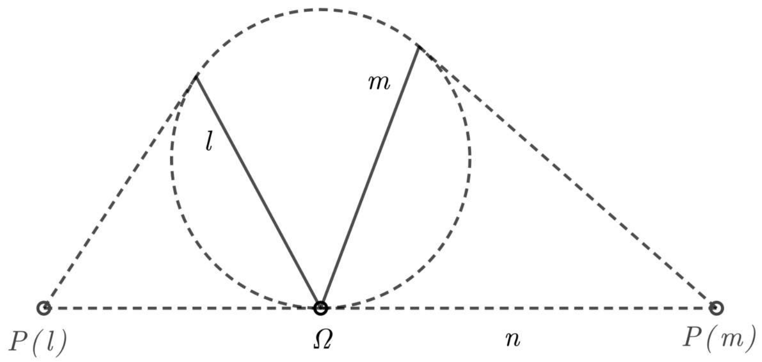

Figure 3.12: Common perpendicular "at infinity" between asymptotic parallels

and we let him walk around the fixed point, always at a constant distance from that point, the result would be hyperbolic circle regardless of how that circle would appear to the observer outside the disk. Considering how distance is defined, it should not come as a surprise that hyperbolic circle does not quite resemble the Euclidean one. The *Figure 3.13* shows a circle with centre C and radius $\overline{AB}$. As we can see, it takes an oval shape, and the centre is not where we would expect it to be. Even though it does not seem that way, the centre C is equidistant to every point on the circumference. This is also an opportunity to observe how a segment of some constant hyperbolic length expands or contracts in Euclidean sense with respect to its position in the disk. As it gets nearer to the boundary of the disk, its Euclidean length will decrease significantly. The only occasion when hyperbolic circle agrees with Euclidean one is when its centre coincides with the centre of the absolute circle.

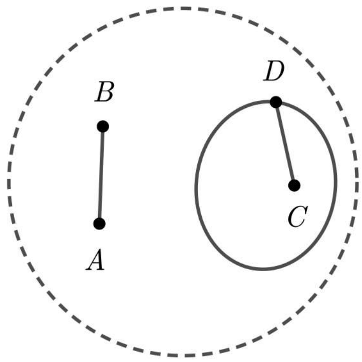

Figure 3.13: Circle in the Beltrami-Klein disk model

3.3 The Poincaré disk model

In 1882, a French mathematician Henri Poincaré (1854–1912) studied two models of hyperbolic geometry that today carry his name: the Poincaré disk and the Poincaré half-plane model. We will first deal with the disk model as it naturally comes after the Beltrami-Klein disk with whom it shares many similarities. Despite the resemblance between them, the Poincaré disk has a big advantage over the Beltrami-Klein disk. It preserves Euclidean angles, and we say that it is *conformal*. The hyperbolic angle between two intersecting hyperbolic lines is the same as the Euclidean angle between those lines. This makes Poincaré disk more suitable to work with, and we can think of it as of an improved version of the Beltrami-Klein disk model.

The hyperbolic plane and points are represented in the same way in both of the models. The hyperbolic plane again resides in the interior of the absolute circle that is fixed in the Euclidean plane. Hyperbolic points are Euclidean points lying inside the absolute circle, points on the boundary are ideal points, and those that lie outside the circle are ultra-ideal points. Definition of lines is what makes the difference between the two disk models. There are two kinds of lines in the Poincaré disk, and they are illustrated in *Figure 3.14*. The hyperbolic line is either a Euclidean diameter of the absolute circle or an arc of a circle orthogonal to the absolute circle. Endpoints on the boundary are not part of the line. Two circles are said to be orthogonal if the tangents at the points of intersection are perpendicular. Note that a diameter of the circle also meets the boundary at right angles, and it can be thought of as an arc of a circle of infinite radius.

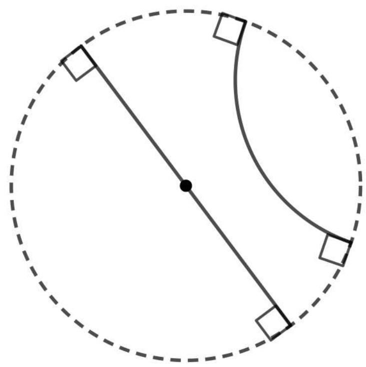

Figure 3.14: Two kinds of lines in the Poincaré disk

Now we need to verify that under this terminology all of the postulates are satisfied. To begin with the first one, let us consider two hyperbolic points in the disk. If these points are collinear, in the Euclidean sense, with the centre of the absolute circle, then they lie on the unique hyperbolic line of the first type. If these points are non-collinear with the centre, then there exists a unique circle that passes through the two points and is orthogonal to the absolute circle (this follows from a theorem of Euclidean geometry, the proof of which can be found in [14, p. 279]). The arc of this circle that lies within the absolute circle is the unique hyperbolic line incident with the given points. Therefore, the first postulate is fulfilled.

With all the tools that we have developed in the previous section, the verification of the second postulate is just as easy. Since points on the boundary are considered to be infinitely far away, all hyperbolic lines extend to infinity. This is supported by metric very similar to that of the Beltrami-Klein model. The proof that the following expression satisfies all metric conditions is analogous to the proof of *Theorem 3.1.*

Definition 3.6. *Let A and B be points inside the absolute circle, and let Ω and Ψ be the ideal points associated with hyperbolic line AB. Then the length $d(A,B)$ in the Poincaré disk model is defined by*

$$d(A,B) = |\ln(AB, \Omega\Psi)| . \tag{5}$$

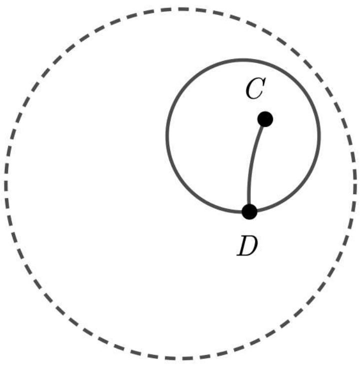

Figure 3.15: Circle in the Poincaré disk model

Our next concern is *Postulate 3.* When a set of all points equidistant from a given point is constructed in Poincaré disk, the result is Euclidean circle but with the centre that differs from the Euclidean centre (except when it coincides with the centre of the absolute circle). A circle with the centre C and the radius $\overline{CD}$ is constructed in *Figure 3.15*.

Considering that the Poincaré disk is conformal, *Postulate 4* is immediately satisfied. Recall that the angle between two intersecting arcs is determined by the measure of the angle formed by the tangents to the arcs at the intersecting point. The *Figure 3.16* shows two pairs of perpendicular lines with associated tangents at the point of intersection.

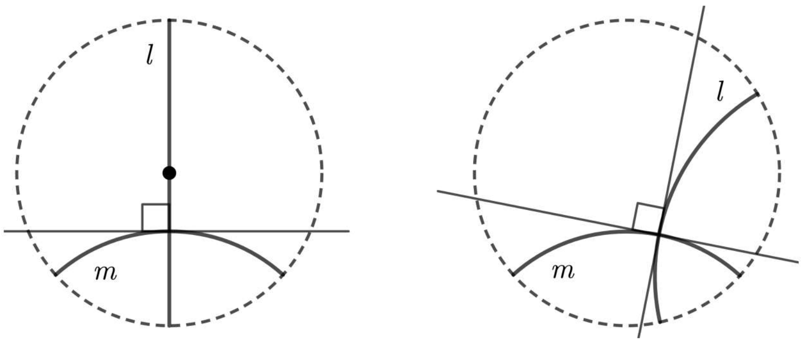

Figure 3.16: Perpendicular lines in the Poincaré disk model

Having confirmed the first four postulates, we can now turn to parallels, our principal subject of interest. In *Figure 3.17*, we have a line l and a point P not on l. There are three different lines m, n and o passing through P, and the task is to determine the nature of these lines with respect to l. The arrangement of lines is much the same as in the Beltrami-Klein disk in *Figure 3.7*. While l and m are intersecting lines, l and o share no common points, and they are divergently parallel. Since Euclidean circles associated with lines l and n intersect at the point on the boundary of the absolute circle, lines l and n are asymptotically parallel. Bearing in mind that a line has two ideal points, there are exactly two asymptotic parallels to l through the point P. On the other hand, through the same point, it is possible to construct infinitely many divergently parallel lines to l.

Among the models of hyperbolic geometry, the Poincaré disk is in a favourable position when it comes to illustrating various geometric objects and relationships between them. In comparison to the Beltrami-Klein disk, some of the lines in the Poincaré model no longer seem straight, but they ensure that the angles are represented accurately. Due to this, the Poincaré disk provides a great visualisation of different concepts from the theory of hyperbolic geometry developed in the previous chapter. We proceed to demonstrate several fragments of this matter. We will also

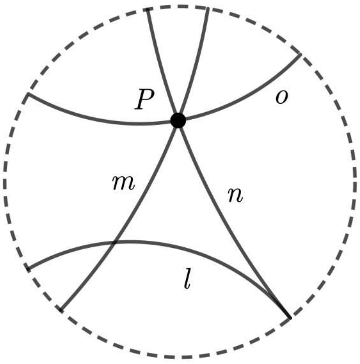

Figure 3.17: Lines in the Poincaré disk model

use this as an opportunity to summarise all the main results.

To remind ourselves of the properties of parallels, let us consider *Figure 3.18* that shows divergent parallels l and m with their unique common perpendicular n. The diagram is in agreement with theorems that we have proved earlier, and it suggests that parallels are not equidistant at every point. They seem to be closest to each other at points of intersections with the common perpendicular, and then they appear to diverge from one another on either side of the perpendicular.

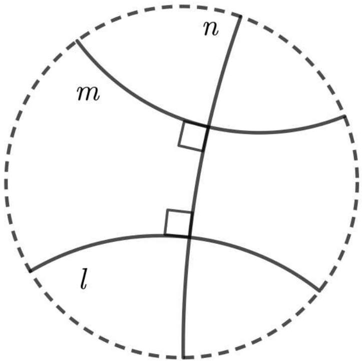

Figure 3.18: Divergent parallels with their common perpendicular

Many theorems regarding divergent parallels were proved with the help of Saccheri quadrilateral. This peculiar object is shown in *Figure 3.19*. All characteristics are neatly portrayed and can be easily recognised. Arms $\overline{AD}$ and $\overline{BC}$ are equal and perpendicular to the base $\overline{AB}$. The summit $\overline{CD}$ is parallel to the base, and the summit angles $\angle BCD$ and $\angle ADC$ are equal and acute. Furthermore, the line PS joining the midpoints of the base and the summit is perpendicular to both.

This indicates that PS is also the unique common perpendicular between divergent parallels AB and CD. In spite of the fact that the distance in the disk is distorted and that we should not rely on the diagram when comparing lengths, in this case the diagram is not wrong, and the summit is indeed longer than the base, just as the arms are greater than $\overline{PS}$. When we divide the Saccheri quadrilateral along this perpendicular PS, we get two congruent Lambert quadrilaterals $ASPD$ and $BSPC$. In each Lambert quadrilateral, three of its angles are right angles, while the fourth one is acute.

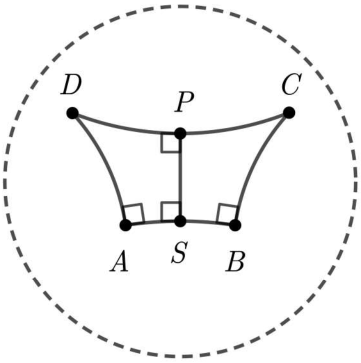

Figure 3.19: Saccheri quadrilateral in the Poincaré disk

A line in the disk models can be denoted by two hyperbolic points, by two ideal points associated with it, or by a combination of one hyperbolic point and one ideal point. In *Figure 3.20*, a line $\Psi\Omega$ is given where Ψ and Ω are its ideal points. Through the point P, there are two asymptotical parallels to the given line, $P\Omega$ and $P\Psi$. Both of them approach the given line asymptotically as we move towards the ideal points they have in common, and they diverge in the other direction. If S is the foot of the perpendicular from P to $\Psi\Omega$, then angles formed by PS and each asymptotic parallel are angles of parallelism. We know that these angles are acute and equal.

Arcs $P\Omega$ and $\Psi\Omega$ are both orthogonal to the absolute circle at the point Ω. Therefore, they share the tangent at the point of intersection, which implies that the angle between them is zero. Same applies to the angle between arcs $\Psi\Omega$ and $P\Psi$. Since all three angles of triangles $\triangle PS\Psi$ and $\triangle PS\Omega$ are equal, these singly asymptotic triangles are congruent. They surely do not seem congruent, but we need to take into account that the distance in the disk is seriously distorted. In the same diagram we can also find doubly asymptotical triangle $\triangle P\Psi\Omega$. It is clear that these triangles have a positive defect, that is, the sum of the angles in each of them is less than two right angles.

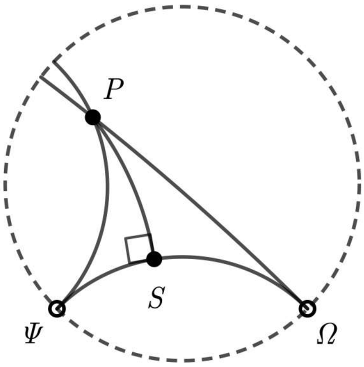

Figure 3.20: Asymptotic parallels in the Poincaré disk

A triply asymptotic triangle is illustrated in *Figure 3.21.* All three vertices are at the boundary of the absolute circle, and the sum of its angles is zero. Accordingly, the defect of this triangle reaches its maximum value, and so does the area. As pointed out earlier, the triply asymptotic triangle is a triangle of the largest possible area and the smallest possible angle sum. Another striking result is that all triply asymptotic triangles are congruent.

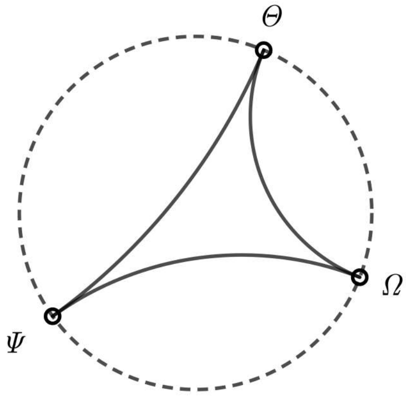

Figure 3.21: Triply asymptotic triangle in the Poincaré disk

Note that the asymptotic triangle is not an actual triangle in the model, because its ideal vertices are not part of the hyperbolic plane. However, it can be approximated by the legitimate triangle whose angles may be as close as desired to zero. Angles in the hyperbolic triangles tend to zero as their vertices approach infinity which in the disk models is represented by the boundary of the absolute circle.

Before we continue with the next model, we take a moment to make a brief digression on the subject worth of mentioning. The hyperbolic geometry was interesting not only to mathematicians, but it also caught the attention of several artists. The most significant among them was a Dutch graphic artist Maurits Cornelis Escher (1898–1972). Inspired by the Poincaré disk, in 1958 he created *Circle Limit I*, the first out of four figures of a *Circle Limit* series, in which he produced infinitely repeated patterns placed in a disk. The resemblance of the *Circle Limit I* with the Poincaré disk is evident (see *Figure 3.22*). By following the spines of the fish, we get either a Euclidean straight line through the centre of the disk, or a Euclidean arc of a circle orthogonal to the boundary. Although the fish seem smaller as they approach the boundary, all the black fish are the same size, and so are the white fish.

In 1959, Escher recreated the first piece and made it more delicate. In this improved version, called the *Circle Limit III*, a white line links the backbones of the fish of the same colour, and the fish swim one after another, head to tail, in one direction. While more appealing than the *Circle Limit I*, a closer examination reveals that white lines in this disk do not intersect the boundary at the right angles, and therefore they do not represent hyperbolic lines at all. Nevertheless, this does not diminish the value of Escher's work. He created many more marvellous art pieces based on different mathematical concepts and showed what an astonishing art can be produced when the passion for mathematics is combined with creativity.

Figure 3.22: *Circle Limit I* and *Circle Limit III*, respectively from left to right

3.4 The Poincaré half-plane model

Another conformal model of hyperbolic geometry due to Poincaré is the half-plane model. It is perhaps the most commonly used model, not because it provides better visualisation than the previous disk models, but rather because it is more appropriate for various calculations. It has a broad application, especially in complex analysis.

To develop the half-plane model, in the Euclidean plane we fix a line instead of a circle. This line is called the absolute line. Then we choose one of the two Euclidean half-planes determined by the absolute line to represent hyperbolic plane. For convenience, in all our diagrams the hyperbolic plane will be the upper half-plane associated with the horizontal absolute line. Hyperbolic points are Euclidean points lying in the chosen half-plane, not including the points on the absolute line which are called ideal points. A hyperbolic line is again interpreted in two different ways. It is either a Euclidean ray emanating from a point on the absolute line and perpendicular to the line, or a Euclidean semicircle with centre on the absolute line. Note that semicircles are orthogonal to the absolute line as well. These two types of lines are illustrated in *Figure 3.23*.

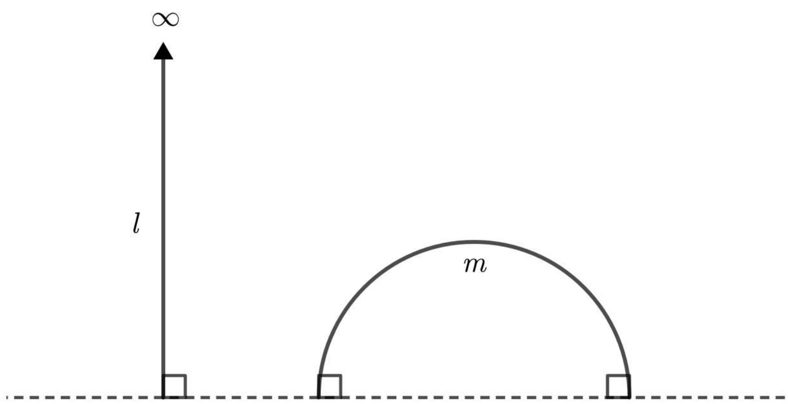

Figure 3.23: Two kinds of lines in the Poincaré half-plane

In addition to the ideal points on the absolute line, one ideal point is at infinity, associated with the direction perpendicular to the absolute line. In other words, it represents the other "end" of every vertical ray. We denote this ideal point by Z, in order to avoid confusion with ideal points on the absolute line. Considering that all vertical lines share the same ideal point Z, they are all asymptotically parallel to one another. An example are lines n and q in *Figure 3.24*. Line n is also one of the two asymptotic parallels to the line l through the point P. Another line through P that shares with l an ideal point on the absolute line is the line m, while line o through P has no points in common with l and it is one of its infinitely many divergent parallels through P. Hyperbolic lines intersect if they intersect in the Euclidean sense. One pair of intersecting lines are o and q.

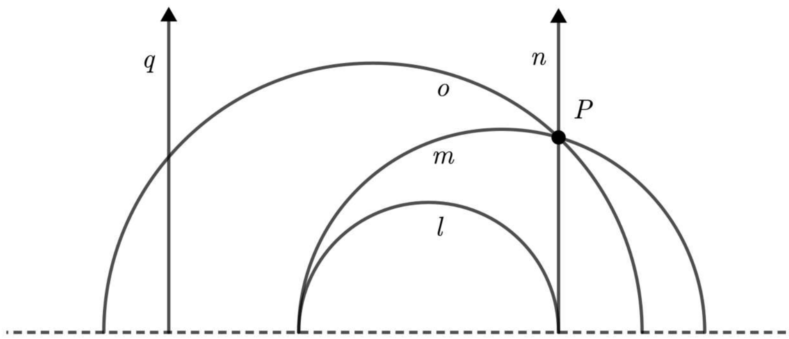

Figure 3.24: Lines in the Poincaré half-plane

In *Figure 3.24*, lines l, m and n form a doubly asymptotic triangle. One could easily fail to notice that in the same diagram there exists a singly asymptotic triangle as well. It is formed by lines q, o and n. Although it does not appear as a triangle, we should not forget that the point Z at infinity is thought to be a mutual "endpoint" to lines q and n. *Figure 3.25* shows two different kinds of triply asymptotic triangles. In the first one, all ideal vertices Θ, Λ and Σ are on the absolute line, while in the asymptotic triangle next to it, vertices Ψ and Ω belong to the absolute line, but the third vertex is at the ideal point Z at infinity.

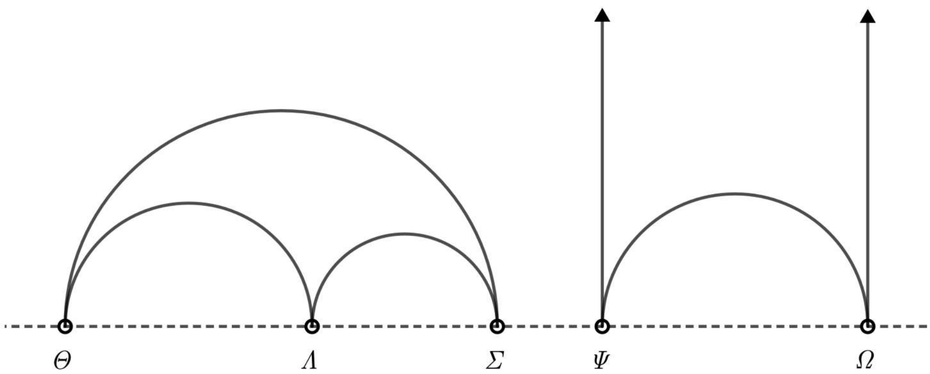

Figure 3.25: Triply asymptotic triangles in the Poincaré half-plane

Since Poincaré half-plane is essentially closely related to the Poincaré disk, there is no need to go through all of the postulates. It is enough to briefly explain and justify only the first two. Verification of the remaining three postulates is trivial.

To confirm that for any two given points A and B there exists a unique hyperbolic line, we drop a Euclidean perpendicular from one of the given points to the absolute line. If another given point lies on that perpendicular then the perpendicular is the unique hyperbolic line containing both given points. If that is not the case, then Euclidean perpendicular bisector of Euclidean segment $\overline{AB}$ intersects absolute line in a point C that is equidistant in Euclidean sense from both A and B. The semicircle with centre C is the unique hyperbolic line passing through A and B.

Infinity of lines follows from the metric. Distance between two points A and B is defined in the same way as in the Poincaré disk (*Definition 3.6*). When the formula is applied to a vertical line, we take one of the ideal points, say Ψ, as the point at infinity, and the cross-ratio $(AB, \Omega\Psi)$ is reduced to the ratio $(AB, \Omega) = \frac{(A\Omega)}{(B\Omega)}$.

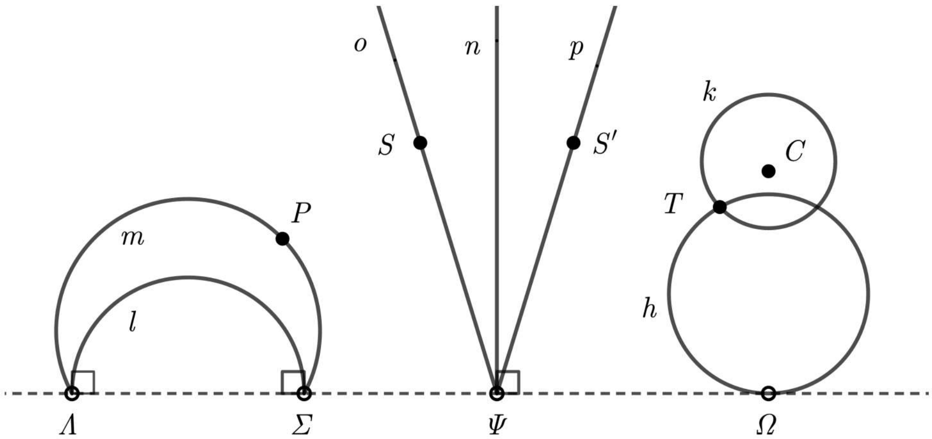

Figure 3.26: Different lines and curves in the Poincaré half-plane model

Now we turn our attention to *Figure 3.26*. There are several Euclidean circles, circular arcs and rays, and we have not assigned the meaning to some of them yet. Semicircle l and ray n are orthogonal to the absolute line, so they represent the hyperbolic lines. We are also familiar with the interpretation of the Euclidean circle k. Since it is positioned entirely within the hyperbolic plane, it serves as a hyperbolic circle. Euclidean circle h, on the other hand, contains an ideal point and cannot be classified as a hyperbolic circle, just as circular arc m, and rays o and p are not orthogonal to the absolute line and cannot be identified as hyperbolic lines. However, none of them is meaningless, and our next goal is to describe their significance and place them in the context of the hyperbolic geometry.

One advantage of Euclidean parallel lines over hyperbolic parallels is that they are equidistant. One would like to develop a similar notion of this important and useful property in the hyperbolic geometry. Following the notation of *Figure 3.26*, let l be a given line associated with ideal points Λ and Σ on the absolute line, and let P be a point not on l. If we construct a curve through P that is at every point equidistant to l, the result is circular arc passing through Λ, P and Σ. Even though it might not seem that way, each point of circular arc m in *Figure 3.26* is indeed at the same hyperbolic distance from the line l. It does not come as a surprise that the curve m is called the *equidistant curve*, or *hypercycle*.

For each line and a given distance, there are two equidistant curves, one on each side of the line, forming equal angles with the absolute line at both ideal points associated with the given line. Note that the equidistant curve and the absolute line do not enclose the right angles. Furthermore, it can be proved that any hyperbolic line perpendicular to l is also perpendicular to the equidistant curve. In case of the vertical type of the hyperbolic line in the Poincaré half-plane model, equidistant curves are rays emanating from the ideal point of the given line. In *Figure 3.26* lines o and p are equidistant curves at the same distance from the line n. An interesting fact is that we have already stumbled upon equidistant curves earlier; the fish backbones in Escher's *Circle Limit III* represent equidistant curves.

It remains to determine the meaning of the Euclidean circle h in *Figure 3.26*. For this purpose, we consider the hyperbolic circle k with centre C through the fixed point T. As previously mentioned, a line in Euclidean geometry can be considered as a circle of infinite radius. We will now observe what curve is obtained in hyperbolic geometry when the centre of a circle tends to infinity.

Due to the distorted distance in the model, the hyperbolic centre of circle k does not coincide with the Euclidean centre. If C' denotes the Euclidean centre of circle k, and Ω denotes the foot of the Euclidean perpendicular from C' to the absolute line, then it can be shown that the hyperbolic centre C lies between Ω and C' on this perpendicular (on the portion inside k). In other words, the hyperbolic centre appears slightly closer to the absolute line than the Euclidean centre. Now suppose that we move the centre C along the perpendicular $C'\Omega$ towards the absolute line. Since point T is fixed, as the centre approaches the infinity, the radius of k approaches infinity as well. The limiting position of circle k is the new curve, denoted by h in *Figure 3.26*, with hyperbolic centre at ideal point Ω. Containing an ideal point, it can no longer be a hyperbolic circle. Instead, it is called the *horocycle*.

In the Poincaré half-plane model, a horocycle takes the form of either a Euclidean circle tangent to the absolute line or a Euclidean line parallel with the absolute line, depending on whether the centre of the horocycle is an ideal point at the absolute line or at Z at infinity. Every hyperbolic line passing through the hyperbolic centre of the horocycle is perpendicular to that horocycle. This property is illustrated in *Figure 3.27*, on horocycle h with hyperbolic centre Ω. Since any hyperbolic line associated with the ideal point Ω is orthogonal to the Euclidean circle h at Ω, it is necessarily perpendicular to h at the other point of intersection. The *Figure 3.27* also shows concentric horocycles e and f. They are both tangent to the absolute line at the ideal point Ψ. Even though f seems smaller in size than e, each one of them is infinite in length.

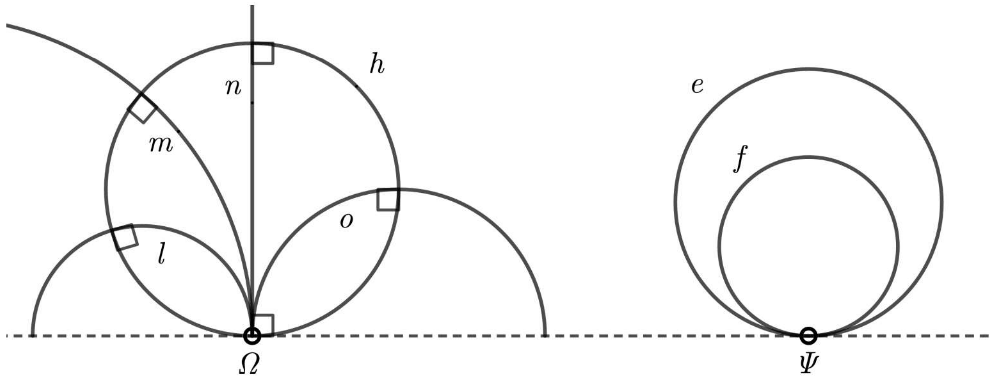

Figure 3.27: Horocycles in the Poincaré half-plane model

As we have seen, a Euclidean circle can have several roles in the Poincaré half-plane, according to its position relative to the absolute line. It could represent a hyperbolic line, hyperbolic circle, equidistant curve or horocycle.

We have illustrated the equidistant curves and horocycles within Poincaré half-plane model, but it is important to emphasize that they belong to the hyperbolic axiomatic theory. Therefore, they are model-independent and exist in every model of hyperbolic geometry.

3.5 The hemisphere

The hemisphere is rarely used as a model itself, but rather as a tool to establish transformations between other models. The Beltrami-Klein disk and the two Poincaré models seem to be, in their essence, fairly similar. We will show that there exists an isomorphism from one model onto the other. Two models are said to be isomorphic if a one-to-one correspondence can be set up between their objects, while the relations between objects are preserved.

The isomorphism between the Beltrami-Klein disk and the Poincaré disk is illustrated in *Figure 3.28.* We start with the sphere in the Euclidean three-dimensional space, with N denoting the north pole of the sphere, and S denoting diametrically opposite south pole. Let the Beltrami-Klein disk be placed in the plane tangent to the sphere at the point S, in such a way that the centre of the absolute circle is at S, and the absolute circle is of the same radius as the sphere. Now we project orthogonally the entire Beltrami-Klein disk onto the sphere. The absolute circle projects to the equator of the sphere, and the points within the disk are sent to the points on the southern hemisphere.

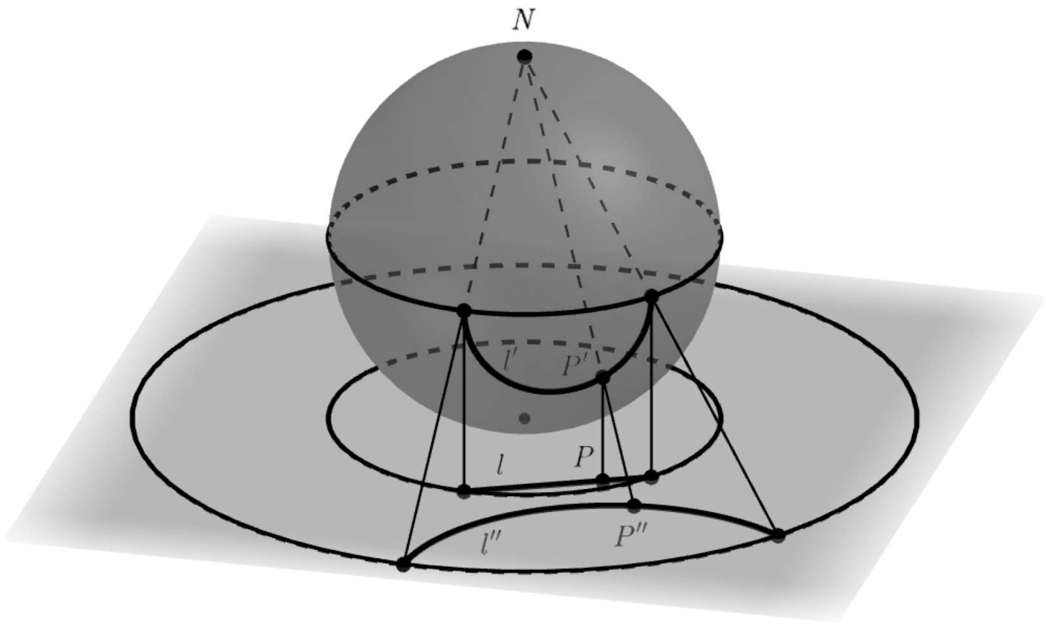

Figure 3.28: An isomorphism from the Beltrami-Klein disk to the Poincaré disk

The hemisphere represents another model of hyperbolic geometry. To determine how the hyperbolic line is defined in this model, it is sufficient to observe the image of the hyperbolic line from the Beltrami-Klein disk under the projection. If l is an open chord of the absolute circle, then the vertical plane which intersects the disk along the given line meets the hemisphere in a semicircle l' orthogonal to the equator of the sphere. Endpoints of the semicircle (points on the equator) are not part of the hyperbolic line.

Having executed the initial step, we can now proceed to the second part of the transformation. To obtain the Poincaré disk model, we project stereographically the southern hemisphere from the north pole N onto the original plane. The equator of the sphere projects to the circle greater than the absolute circle of Beltrami-Klein disk, and the southern hemisphere maps to its interior. The semicircle of the hemisphere l' projects to l'' which is either a diameter or a circular arc orthogonal to the boundary. The former occurs when the line l' passes through the south pole of the sphere. Note that this whole process is reversible. Thus, under the composition of orthogonal and stereographic projection, there exists a one-to-one correspondence between the points P and P'' of the Beltrami-Klein and the Poincaré disk model, and a one-to-one correspondence between the lines l and l'' of the two models, such that P lies on l if and only if P'' lies on l''. This isomorphism also preserves the congruence of segments and angles.

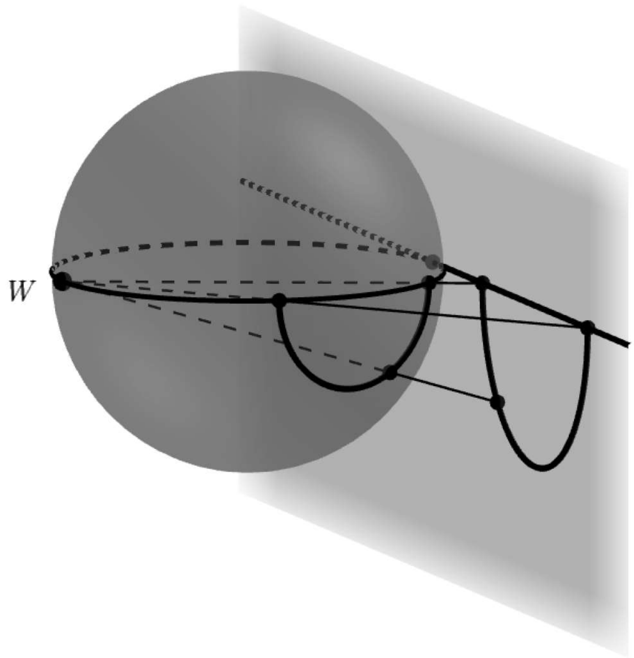

Figure 3.29: An isomorphism from the hemisphere to the Poincaré half-plane

Similar transformation can be established with the Poincaré half-plane model (see *Figure 3.29*). This time the southern hemisphere is projected stereographically from the point on the equator, denoted by W, to the plane tangent to the sphere at the point diametrically opposite to W. The equator is mapped to the horizontal absolute line, and the points of the southern hemisphere are sent to the points of the lower half-plane determined by the absolute line. A semicircle of the hemisphere becomes either semicircle centred at the absolute line or a vertical ray perpendicular to the absolute line. The point W got lost in the stereographic projection, but we know that it is mapped to the point Z at infinity since all semicircles passing through W map to vertical rays passing through Z.

Therefore, we have found an isomorphism between the four models of hyperbolic geometry. In fact, it is possible to prove that all models of hyperbolic geometry are mutually isomorphic. An axiomatic system with this property is said to be *categorical*. This is one of the characteristics that hyperbolic geometry shares with Euclidean geometry. It implies that there is basically only one distinct representation for the axiomatic system. Interpretation of objects within the system may vary from model to model, but they essentially remain "the same". Consequently, we have the freedom to move between the models, and we are able to view objects and relations of hyperbolic geometry from different perspectives, while nothing significantly changes.

4 Geometry of the physical world

Having developed the models that support the theory, it is now indisputable that hyperbolic axiomatic system is just as consistent and accurate as Euclidean geometry. Nevertheless, given that we do not encounter hyperbolic geometry in everyday life, one may easily draw the wrong conclusion that the entire analysis that we have done so far serves only as an amusing intellectual game. It would be misleading to think of geometry as the branch of mathematics in which we start with a set of arbitrarily selected axioms and then logically deduce different results from them just for the sake of entertainment. While it is possible to practice mathematics solely for the beauty of mathematics itself, it is highly unlikely that any professional mathematician would devote her or his life to a field that neither proves fruitful nor has a higher purpose. An axiomatic system that does not lead anywhere eventually becomes abandoned and forgotten.

Hyperbolic geometry, on the other hand, did not receive the attention it deserved during the lifetime of Lobachevsky, Bolyai and Gauss, but ultimately, its great potential was recognised, and the true value came to light. Perhaps the most powerful application is the one related to the geometry of the universe. Being one of the oldest branches of mathematics, geometry has always been interconnected with the physical world. Ever since ancient times, it was a reliable source of information about the real spatial relationships. Conventional wisdom has it that the world in which we live is best described by Euclidean geometry. However, how can we be sure that the world is not, in fact, hyperbolic or spherical? After we accept the idea that Euclidean geometry is only one of the several candidates, a major question that arises is which of the possible geometries most accurately represents the physical space around us. This seemingly simple question actually deals with a complex matter and is quite hard to answer. The phrase "there is more than meets the eye" has never been more appropriate than in this context.

One feasible approach to the problem would be to find a physical triangle and precisely measure its angles. This way, we could determine whether the sum of the angles is in agreement with hyperbolic, Euclidean or spherical geometry. The first known experiment of this kind was conducted by Gauss, although we cannot claim with certainty that he had any intention of finding the evidence that the geometry of the space was non-Euclidean. In 1820 Gauss was in charge of mapping Hannover, a state in Germany. In order to do so, he invented a measuring device which relied on reflected sun rays. Gauss assumed that the light travels in a straight line, and he observed a huge triangle formed by light rays with vertices at the peaks of three mountains. Results that he gathered were inconclusive; deviation of the angle sum from two right angles was within experimental error. However, this should not be taken as confirmation of Euclidean geometry. As a matter of fact, Euclidean geometry cannot be empirically confirmed because no matter how precise instruments we have, and how close the sum of the angles is to two right angles, the experimental error can never be entirely eliminated. With this kind of research, we can only prove that the space is non-Euclidean, provided that the result significantly differs from the Euclidean angle sum.

Furthermore, we mentioned earlier (in *Section 2.3*) that small hyperbolic triangles are indistinguishable from Euclidean ones, but "small" is a relative term. It is not specified just how large triangle should be for its defect to be measurable. Even though the Gauss's triangle was enormous (its longest side was over 100 km long), perhaps it was still too minor for any considerable variation. In more recent times, similar experiments were conducted on far greater triangles with distinct stars as vertices. The outcome of every experiment was the same; no significant deviation has been obtained. But even these tremendous triangles can be considered small compared to the overall size of the universe.

A different route to determine the nature of the geometry of the universe is to detect its curvature. Generally, the most effective way to recognise the curvature of some space is by observing that space from the outside. Unfortunately, this approach is clearly impossible when it comes to the universe. We might notice that our situation is analogous to that of the two-dimensional inhabitant of the Beltrami-Klein disk in *Figure 3.8*. Due to the inability to externally view the world in which we live, we have no choice but to seek an alternative method to comprehend its true characteristics.

Major scientific contribution in this area was made by Albert Einstein (1879–1955). According to his theory of general relativity, the gravity of a massive object in the universe bends the space in the vicinity of that object. Distorted space causes the light rays affected by a gravitational field to no longer travel in a straight line, even though we perceive them as straight. Therefore, the curvature of the universe varies from one part of the universe to another. For instance, the curvature in the vicinity of the sun is greater than the curvature near the Earth. As a result, when describing the space, all three geometries need to be considered. While some portions of the universe are best approximated by Euclidean geometry, for other regions hyperbolic and spherical geometry might be more appropriate.

Nevertheless, scientists believe that there is one overall curvature of the entire universe. This conjecture sounds less strange when we realise that much the same circumstances apply to the surface of the Earth. While Earth's local curvature differs remarkably depending on the location (e.g. the Himalayas versus the Pannonian Plane), we know that the global shape of the Earth is spherical. However, the overall comprehensive geometry that would describe the universe on a full-scale remains an open question and has yet to be found.

It is interesting that the general theory of relativity, which brought revolutionary changes in physics, was established on the grounds of the non-Euclidean geometry. Einstein admitted that he had issues of expressing his ideas mathematically, and according to his words, he would not have been able to develop the general relativity if it had not been for the discoveries in the field of geometry [6, p. 373]. It is rather fascinating how the question of the Parallel postulate, which dates back nearly 2000 years, has such a profound impact on contemporary science. Throughout its long history, the subject of parallels intrigued many mathematicians and inspired their work, but probably none of them would have ever guessed the far-reaching consequences of this initially abstract matter. Besides the application in physics, which we have barely scratched, the non-Euclidean geometry also has a broad implementation in numerous branches of mathematics. Although it has already made a large contribution in various areas, the full potential of non-Euclidean geometry is still far from being exhausted. The fact that it remains to be a topic of interest to this day proves that non-Euclidean geometry stands alongside Euclidean geometry as its rightful companion. It is unquestionable that both geometries are equally worthy, and this was best put into words by Poincaré: "One geometry cannot be more true than another; it can only be more convenient." [6, p. 374]

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Summary

Geometry based on the five postulates proposed by Euclid was considered the only geometry possible for more than two millennia. It remained unchallenged until the early 19th century, when mathematicians Lobachevsky, Bolyai and Gauss, independently discovered that by modifying the Parallel postulate, they were able to develop an axiomatic system that significantly differed from the Euclidean geometry, but was equally consistent. At the beginning of the thesis, we follow a concise historical course of events that preceded the revolutionary breakthrough of the hyperbolic geometry. Afterwards, we state results unique to this axiomatic system, which mostly disagree with our intuitive experience. The underlying subject of the study are parallels, and we classify them according to their characteristics on divergent and asymptotic parallels. Then we introduce the most common models of hyperbolic geometry, on which we demonstrate the previously proven properties of two kinds of parallels, and various abstract concepts of hyperbolic geometry. The following models are observed: the pseudosphere, Beltrami-Klein disk, Poincaré disk, Poincaré half-plane and hemisphere. Using the hemisphere model, we establish transformations between the Beltrami-Klein and Poincaré models. Finally, we briefly discuss the connection of the hyperbolic geometry with the physical world and its role in Einstien's theory of general relativity.

Keywords: hyperbolic geometry, Parallel postulate, Saccheri, Lobachevsky, Bolyai, Gauss, divergent parallels, asymptotic parallels, pseudosphere, Beltrami-Klein disk, Poincaré disk, Poincaré half-plane, hemisphere

Sažetak

Geometrija utemeljena na Euklidovih pet postulata smatrala se jedinom mogućom geometrijom dulje od dva tisućljeća. Dovedena je u pitanje tek početkom 19. stoljeća, kada su matematičari Lobachevsky, Bolyai i Gauss, neovisno jedan o drugome, spoznali da modifikacijom postulata o paralelama mogu razviti aksiomatski sustav značajno različit od euklidske geometrije, ali jednako konzistentan. Na početku rada pratimo sažet povijesni tijek događaja koji su prethodili revolucionarnom otkriću hiperbolne geometrije. Potom navodimo rezultate jedinstvene ovom aksiomatskom sustavu koji su nerijetko u sukobu s našim intuitivnim iskustvom. Inicijalni predmet proučavanja su paralele i njih klasificiramo prema njihovim karakteristikama na divergentne i asimptotske. U nastavku konstruiramo najčešće modele hiperbolne geometrije na kojima demonstriramo prethodno dokazana svojstva dvaju tipova paralela, te ostale apstraktne koncepte hiperbolne geometrije. Promatrani su sljedeći modeli: pseudokugla, Beltrami-Klein disk, Poincaré disk, Poincaré poluravnina, te polukugla. Pomoću modela polukugle prikazane su transformacije između Beltrami-Klein i Poincaré modela. Naposljetku smo se dotaknuli povezanosti hiperbolne geometrije s fizičkim svijetom, te njenom ulogom u Einsteinovoj općoj teoriji relativnosti.

Ključne riječi: hiperbolna geometrija, postulat o paralelama, Saccheri, Lobachevsky, Bolyai, Gauss, divergentne paralele, asimptotske paralele, pseudokugla, Beltrami-Klein disk, Poincaré disk, Poincaré poluravnina, polukugla

Biography

I was born on 23 April 1994, in Našice, Croatia. My educational journey started in 2001 at the elementary school “Matija Gubec” in Magadenovac, and continued in 2009 at the high school “Isidor Kršnjavi” in Našice. Upon graduating from the gymnasium in 2013, I enrolled in the Integrated undergraduate and graduate university study programme in mathematics and computer science at the Department of mathematics of the Josip Juraj Strossmayer University of Osijek.
During my studies, I volunteered at association “Dokkica - Children’s creative house Osijek”, where I held individual tuitions in mathematics to a student with learning difficulties. In 2019, I took part in Erasmus+ student exchange programme, and from February to June, I attended Charles University in Prague. Throughout this academic period, I completed a supervised teaching practice at bilingual primary and secondary school “EDUCAnet Praha”. Apart from the internship, I also participated in the project “Europe meets School”, in which I took the role of an ambassador of my home country in partner elementary school of Charles University.