Runge Kutta metode za numeričko rješavanje diferencijalnih jednadžbi

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Voditelj: prof. dr. sc. Kristian Sabo

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Runge Kutta methods for numerical solving differential equations

Sažetak U ovome radu ukratko ćemo se upoznati s Runge Kutta metodama za numeričko rješavanje diferencijalnih jednadžbi. U uvodnom dijelu rada su opisane Runge Kutta metode i dan je primjer za rješavanje Runge Kutta drugog reda. Glavni dio rada bavi se izvodom Runge Kutta metode četvrtog reda, a na kraju su dani pripadni primjeri.

Ključne riječi Runge Kutta metode, diferencijalna jednadžba, izvod

Abstract In this paper we will be introduced to the Runge Kutta methods for numerical solving differential equations. In first part of paper we described Runge Kutta methods and we given example for solving second order Runge Kutta. The main part of the paper refers to proof of the fourth order Runge Kutta, and at the end we given two examples.

Key words Runge Kutta methods, differential equation, the proof

Sadržaj

1 Uvod

Mnogi matematički modeli koji se pojavljuju u različitim primjenama mogu se opisati pomoću običnih diferencijalnih jednadžbi. Vrlo često obične diferencijalne jednadžbe nije moguće riješiti egzaktno te su u tu svrhu razvijene numeričke metode za njihovo približno rješavanje. Jedna od najpoznatijih klasa metoda za numeričko rješavanje običnih diferencijalnih jednadžbi jesu Runge Kutta metode.

U ovome radu razmatramo Runge Kutta metode za numeričko rješavanje običnih diferencijalnih jednadžbi. Rad se sastoji od četiri poglavlja. U drugom poglavlju dajemo opći zapis Runge Kutta metoda te izvodimo eksplicitne formule za Runge Kutta metode prvog i drugog reda. Pokazuje se da Runge Kutta metoda prvog reda odgovara Eulerovoj metodi. Najvažniji dio rada sadržan je u trećem poglavlju u kojem izvodimo eksplicitne formule za Runge Kutta metodu četvrtog reda. U četvrtom poglavlju bavimo se rješavanjem sustava diferencijalnih jednadžbi pomoću Runge Kutta metode. Za svaku od spomenutih metoda dani su ilustrativni primjeri.

2 Runge Kutta metode

Prije samog definiranja Runge Kutta metode razmotrit ćemo zašto nam je uopće potrebno numeričko rješavanje diferencijalnih jednadžbi. Promatramo sljedeći problem (vidi [3]):

Za danu funkciju $f(x,y)$ treba pronaći funkciju $y(x)$, $y:[a,b]\to\mathbb{R}$ gdje je $x\in[x_0,b]$ koja zadovoljava diferencijalnu jednadžbu prvoga reda

$$\frac{dy}{dx}=f(x,y), \tag{1}$$

uz početni uvjet

$$y(x_0)=y_0. \tag{2}$$

Problem (1)-(2) poznat je kao **inicijalni** ili **Cauchyjev problem** (vidi [4]). Postoje slučajevi u kojima se ovakav problem može egzaktno riješiti, ali najčešće nailazimo na problem koji moramo riješiti aproksimativno. Kao primjer jednog takvog problema možemo uzeti Cauchyjevu zadaću $y'=e^{y-1}, y(0)=1$ (vidi[2]). Jedan od načina rješavanja je pomoću Runge Kutta metode koju ćemo opisati u daljnjem tekstu.

2.1 Opći oblik Runge Kutta metode

Da bismo riješili dani Cauchyjev problem potrebno je napraviti subdiviziju segmenta $[a,b]$:

$$a=x_0,x_1,x_2,\ldots,x_n=b.$$

Označimo s $h{=}x_{i+1}-x_i$, $i=0,\ldots,n-1$. Vrijednost y_{i+1} u točki x_{i+1} računamo pomoću poznate vrijednosti y_i u točki x_i. Opći oblik Runge Kutta metode određen je formulom

$$y_{k+1}=y_k+h\sum_{i=1}^{m}c_i k_i^{(k)}$$

gdje je $k=0,\ldots,n-1$, m red metode, a $k_i^{(k)}$ je dan izrazom

$$k_i^{(k)}=f\left(x_k+\alpha_i h, y_k+h\sum_{j=1}^{i-1}\beta_{ij}k_j^{(k)}\right),\ i=1,\ldots,m.$$

2.2 Runge Kutta metoda prvoga reda (RK-1)

Koristeći opći oblik Runge Kutta metode dobivamo

$$y_{k+1}=y_k+hc_1k_1^{(k)}=y_k+hc_1f(x_k,y_k) \tag{3}$$

Nadalje, razvijemo li izraz $y_{k+1}{=}y(x_k+h)$ u Taylorov red oko točke x_k dobivamo

$$y_{k+1}=y(x_k+h)=y(x_k)+hy'(x_k)+O(h^2)=y_k+hf(x_k,y_k)+O(h^2). \tag{4}$$

Usporedbom izraza (3) i (4) vidimo da postoji točno jedna Runge Kutta metoda prvoga reda, a to je upravo Eulerova metoda (vidi [5]).

2.3 Runge Kutta metode drugoga reda (RK-2)

Kao kod Runge Kutta metode prvoga reda koristit ćemo opći oblik pa dobivamo

$$y_{k+1} = y_k + h(c_1 k_1^{(k)} + c_2 k_2^{(k)}),$$

pri čemu su

$$k_1^{(k)} = f(x_k, y_k)$$
$$k_2^{(k)} = f(x_k + \alpha_2 h, y_k + h\beta_{21} k_1^{(k)}).$$

Treba odrediti koeficijente α_2, β_{21}, c_1 i c_2. Pripadni koeficijenti neće biti jednoznačno određeni pa će Runge Kutta metoda drugoga reda biti beskonačno mnogo (vidi [3]). Izvodom bismo dobili da Runge Kutta metode drugoga reda moraju zadovoljavati sljedeće uvjete:

$$c_1 + c_2 = 1$$
$$c_2 \alpha_2 = \frac{1}{2}$$
$$c_2 \beta_{21} = \frac{1}{2}.$$

Najčešća varijanta Runge Kutta metode drugoga reda za koeficijente uzima $\alpha_2 = 1$, $c_2 = \frac{1}{2}$, $c_1 = \frac{1}{2}$, $\beta_{21} = 1$ pa tada dobivamo

$$y_{k+1} = y_k + \frac{h}{2}(k_1^{(k)} + k_2^{(k)})$$
$$k_1^{(k)} = f(x_k, y_k)$$
$$k_2^{(k)} = f(x_k + h, y_k + hf(x_k, y_k))$$

Pripadna metoda naziva se Heuneova metoda. Uzmemo li $\alpha_2 = \frac{1}{2}$, $c_2 = 1$, $c_1 = 0$, $\beta_{21} = \frac{1}{2}$ dobivamo standardnu RK-2 metodu koja je oblika

$$y_{k+1} = y_k + hk_2^{(k)}$$
$$k_1^{(k)} = f(x_k, y_k)$$
$$k_2^{(k)} = f(x_k + \frac{h}{2}, y_k + \frac{1}{2}hf(x_k, y_k)).$$

2.4 Primjer Runge Kutta metode drugog reda

Primjer 2.1.

Pomoću Heuneove metode riješiti jednadžbu $y' = -x^2 y$ *s početnim uvjetom* $y(0) = 2$ *na* $[0,3]$ *uz korak* 0.5.

Rješenje:
Elementi subdivizije segmenta $[0,3]$ glase $x_i = x_0 + ih = 0.5i$, $i = 0, \dots, 6$. Potrebno je odrediti vrijednosti y_{k+1} za $k = 0, \dots, 5$. Rezultate ćemo zapisati u tablicu.

Zapišimo najprije opći oblik Heuneove metode:

$$
\begin{gathered}
y_{k+1} = y_k + \tfrac{h}{2}(k_1^{(k)} + k_2^{(k)}) \\
k_1^{(k)} = f(x_k, y_k) \\
k_2^{(k)} = f(x_k + h, y_k + hf(x_k, y_k))
\end{gathered}
$$

Odredimo sada y_1.

$$
\begin{gathered}
y_1 = y_0 + \tfrac{h}{2}(k_1^{(0)} + k_2^{(0)}) \\
k_1^{(0)} = f(x_0, y_0) = f(0, 2) = 0 \\
k_2^{(0)} = f(x_0 + h, y_0 + hf(x_0, y_0)) = f(0.5, 2) = -0.5
\end{gathered}
$$

Vratimo $k_1^{(0)}$ i $k_2^{(0)}$ u y_1 i dobivamo

$$
y_1 = 2 + \tfrac{0.5}{2}(0 + (-0.5)) = 1.875
$$

Ponovimo postupak kako bi pronašli preostale vrijednosti $y_2, \dots, y_6$

$$
\begin{gathered}
y_2 = y_1 + \tfrac{h}{2}(k_1^{(1)} + k_2^{(1)}) \\
k_1^{(1)} = f(x_1, y_1) = f(0.5, 1.875) = -0.46875 \\
k_2^{(1)} = f(x_1 + h, y_1 + hf(x_1, y_1)) = f(1, 1.640625) = -1.640625 \\
y_2 = 1.34766
\end{gathered}
$$

$$
\begin{gathered}
y_3 = y_2 + \tfrac{h}{2}(k_1^{(2)} + k_2^{(2)}) \\
k_1^{(2)} = f(x_2, y_2) = f(1, 1.34766) = -1.34766 \\
k_2^{(2)} = f(x_2 + h, y_2 + hf(x_2, y_2)) = f(1.5, 0.67383) = -1.51612 \\
y_3 = 0.63172
\end{gathered}
$$

$$
\begin{gathered}
y_4 = y_3 + \tfrac{h}{2}(k_1^{(3)} + k_2^{(3)}) \\
k_1^{(3)} = f(x_3, y_3) = f(1.5, 0.63172) = -1.42137 \\
k_2^{(3)} = f(x_3 + h, y_3 + hf(x_3, y_3)) = f(2, -0.078965) = 0.31586 \\
y_4 = 0.35534
\end{gathered}
$$

$$
\begin{gathered}
y_5 = y_4 + \tfrac{h}{2}(k_1^{(4)} + k_2^{(4)}) \\
k_1^{(4)} = f(x_4, y_4) = f(2, 0.35534) = -1.42136 \\
k_2^{(4)} = f(x_4 + h, y_4 + hf(x_4, y_4)) = f(2.5, -0.35534) = 2.220875 \\
y_5 = 0.55522
\end{gathered}
$$

$$
\begin{gathered}
y_6 = y_5 + \tfrac{h}{2}(k_1^{(5)} + k_2^{(5)}) \\
k_1^{(5)} = f(x_5, y_5) = f(2.5, 0.55522) = -3.470125 \\
k_2^{(5)} = f(x_5 + h, y_5 + hf(x_5, y_5)) = f(3, -1.1798425) = 10.618583 \\
y_6 = 2.34233
\end{gathered}
$$

k	x_k	y_k
0	0	2
1	0.5	1.875
2	1	1.34766
3	1.5	0.63172
4	2	0.35534
5	2.5	0.55522
6	3	**2.34233**

Tablica 1: Cauchyjev problem $y' = -x^2 y$, y(0)=2, $x \in [0,3]$

3 Runge Kutta metode četvrtog reda (RK-4)

Runge Kutta metoda četvrtog reda je oblika

$$y_{k+1} = y_k + h(c_1 k_1^{(k)} + c_2 k_2^{(k)} + c_3 k_3^{(k)} + c_4 k_4^{(k)}), \tag{5}$$

pri čemu su

$$\begin{aligned} k_1^{(k)} &= f(x_k, y_k) \\ k_2^{(k)} &= f(x_k + \alpha_2 h, y_k + h\beta_{21} k_1^{(k)}) \\ k_3^{(k)} &= f(x_k + \alpha_3 h, y_k + h(\beta_{31} k_1^{(k)} + \beta_{32} k_2^{(k)})) \\ k_4^{(k)} &= f(x_k + \alpha_4 h, y_k + h(\beta_{41} k_1^{(k)} + \beta_{42} k_2^{(k)} + \beta_{43} k_3^{(k)})) \end{aligned}$$

za $k = 0, \ldots, n-1$, $c_1, c_2, c_3, c_4 \in \mathbb{R}$, $\alpha_2, \alpha_3, \alpha_4, \beta_{21}, \beta_{31}, \beta_{32}, \beta_{41}, \beta_{42}, \beta_{43} \in \mathbb{R}$.

3.1 Izvod

Razvoj funkcije $y(x_k + h)$ u Taylorov red oko točke x_k glasi:

$$y_{k+1} = y(x_k + h) = y(x_k) + h\, y'(x_k) + \frac{h^2}{2} y''(x_k) + \frac{h^3}{3!} y'''(x_k) + \frac{h^4}{4!} y^{(4)}(x_k) + O(h^5).$$

Znamo da je $f(x,y) = y'(x)$. Potrebno je odrediti derivacije y'', y''' i $y^{(4)}$.

$$
\begin{aligned}
y''(x) &= \frac{d^2y}{dx^2} = \frac{d}{dx}f(x,y) = f_x + f_y f \\
y'''(x) &= \frac{d}{dx}(y''(x)) = f_{xx} + f_{xy}f + (f_{xy} + f_{yy}f)f + f_y(f_x + f_y f) = f_{xx} + 2f_{xy}f + f_{yy}f^2 + f_x f_y + f_y^2 f \\
y^{(4)}(x) &= \frac{d}{dx}(y'''(x)) = f_{xxx} + f_{xxy}f + (2f_{xxy} + 2f_{xyy}f)f + (2f_x + 2f_y f)f_{xy} \\
&\quad + (f_{xyy} + f_{yyy}f)f^2 + 2f(f_x + f_y f)f_{yy} + (f_{xx} + f_{xy}f)f_y + (f_{xy} + f_{yy}f)f_x \\
&\quad + 2f_y(f_{xy} + f_{yy}f)f + (f_x + f_y f)f_y^2 = f_{xxx} + f_{xxy}f + 2f_{xxy}f + 2f_{xyy}f^2 \\
&\quad + 2f_x f_{xy} + 2f_y f_{xy}f + f_{xyy}f^2 + f_{yyy}f^3 + 2f_x f_{yy}f + 2f_y f_{yy}f^2 + f_{xx}f_y \\
&\quad + f_{xy}f_y f + f_{xy}f_x + f_{yy}f_x f + 2f_y f_{xy}f + 2f_y f_{yy}f^2 + f_x f_y^2 + f_y^3 f \\
&= f_{xxx} + 3f_{xxy}f + 3f_{xyy}f^2 + 3f_{xy}f_x + 5f_{xy}f_y f \\
&\quad + f_{yyy}f^3 + 3f_x f_{yy}f + 4f_y f_{yy}f^2 + f_{xx}f_y + f_x f_y^2 + f_y^3 f.
\end{aligned}
$$

Sada imamo:

$$
\begin{aligned}
y_{k+1} &= y_k + hf(x_k, y_k) + \frac{h^2}{2}(f_x + f_y f) \\
&\quad + \frac{h^3}{6}(f_{xx} + 2f_{xy}f + f_{yy}f^2 + f_x f_y + f_y^2 f) \\
&\quad + \frac{h^4}{24}(f_{xxx} + 3f_{xxy}f + 3f_{xyy}f^2 + 3f_{xy}f_x \\
&\quad + 5f_{xy}f_y f + f_{yyy}f^3 + 3f_x f_{yy}f + 4f_y f_{yy}f^2 + f_{xx}f_y + f_x f_y^2 + f_y^3 f)
\end{aligned} \tag{6}
$$

S druge strane, razvijamo u Taylorov red koeficijente $k_2^{(k)}, k_3^{(k)}, k_4^{(k)}$ pri čemu koristimo oznake $\gamma_1 = \beta_{21}k_1^{(k)}, \gamma_2 = \beta_{31}k_1^{(k)} + \beta_{32}k_2^{(k)}, \gamma_3 = \beta_{41}k_1^{(k)} + \beta_{42}k_2^{(k)} + \beta_{43}k_3^{(k)}$:

$$
\begin{aligned}
k_2^{(k)} &= f + h\alpha_2 f_x + h\gamma_1 f_y + \frac{h^2}{2}(\alpha_2 f_x + \gamma_1 f_y)^2 \\
&\quad + \frac{h^3}{6}(\alpha_2 f_x + \gamma_1 f_y)^3 + O(h^4) = f + h\alpha_2 f_x + h\gamma_1 f_y \\
&\quad + \frac{h^2}{2}(\alpha_2^2 f_{xx} + 2\alpha_2\gamma_1 f_{xy} + \gamma_1^2 f_{yy}) \\
&\quad + \frac{h^3}{6}(\alpha_2^3 f_{xxx} + 3\alpha_2^2\gamma_1 f_{xxy} + 3\alpha_2\gamma_1^2 f_{xyy} + \gamma_1^3 f_{yyy}) + O(h^4),
\end{aligned}
$$

$$
\begin{aligned}
k_3^{(k)} &= f + h\alpha_3 f_x + h\gamma_2 f_y + \frac{h^2}{2}(\alpha_3 f_x + \gamma_2 f_y)^2 \\
&\quad + \frac{h^3}{6}(\alpha_3 f_x + \gamma_2 f_y)^3 + O(h^4) = f + h\alpha_3 f_x + h\gamma_2 f_y \\
&\quad + \frac{h^2}{2}(\alpha_3^2 f_{xx} + 2\alpha_3\gamma_2 f_{xy} + \gamma_2^2 f_{yy}) \\
&\quad + \frac{h^3}{6}(\alpha_3^3 f_{xxx} + 3\alpha_3^2\gamma_2 f_{xxy} + 3\alpha_3\gamma_1^2 f_{xyy} + \gamma_2^3 f_{yyy}) + O(h^4),
\end{aligned}
$$

te

$$
\begin{aligned}
k_4^{(k)} &= f + h\alpha_4 f_x + h\gamma_3 f_y + \frac{h^2}{2}(\alpha_4 f_x + \gamma_3 f_y)^2 \\
&+ \frac{h^3}{6}(\alpha_4 f_x + \gamma_3 f_y)^3 + O(h^4) = f + h\alpha_4 f_x + h\gamma_3 f_y \\
&+ \frac{h^2}{2}(\alpha_4^2 f_{xx} + 2\alpha_4\gamma_3 f_{xy} + \gamma_3^2 f_{yy}) \\
&+ \frac{h^3}{6}(\alpha_4^3 f_{xxx} + 3\alpha_4^2\gamma_3 f_{xxy} + 3\alpha_4\gamma_3^2 f_{xyy} + \gamma_3^3 f_{yyy}) + O(h^4).
\end{aligned}
$$

Kada u $y_{k+1} - y_k = h(c_1 k_1^{(k)} + c_2 k_2^{(k)} + c_3 k_3^{(k)} + c_4 k_4^{(k)})$ uvrstimo gore raspisane koeficijente dobivamo

$$
\begin{aligned}
y_{k+1} - y_k = h\big[&c_1 f + c_2(f + h\alpha_2 f_x + h\gamma_1 f_y + \frac{h^2}{2}(\alpha_2^2 f_{xx} + 2\alpha_2\gamma_1 f_{xy} \\
&+ \gamma_1^2 f_{yy}) + \frac{h^3}{6}(\alpha_2^3 f_{xxx} + 3\alpha_2^2\gamma_1 f_{xxy} + 3\alpha_2\gamma_1^2 f_{xyy} \\
&+ \gamma_1^3 f_{yyy}) + O(h^4)) + c_3(f + h\alpha_3 f_x + h\gamma_2 f_y + \frac{h^2}{2}(\alpha_3^2 f_{xx} + 2\alpha_3\gamma_2 f_{xy} \\
&+ \gamma_2^2 f_{yy}) + \frac{h^3}{6}(\alpha_3^3 f_{xxx} + 3\alpha_3^2\gamma_2 f_{xxy} + 3\alpha_3\gamma_1^2 f_{xyy} \\
&+ \gamma_2^3 f_{yyy}) + O(h^4)) + c_4(f + h\alpha_4 f_x + h\gamma_3 f_y + \frac{h^2}{2}(\alpha_4^2 f_{xx} + 2\alpha_4\gamma_3 f_{xy} \\
&+ \gamma_3^2 f_{yy}) + \frac{h^3}{6}(\alpha_4^3 f_{xxx} + 3\alpha_4^2\gamma_3 f_{xxy} + 3\alpha_4\gamma_3^2 f_{xyy} + \gamma_3^3 f_{yyy}) + O(h^4))\big]
\end{aligned}
\tag{7}
$$

Sada uspoređujemo (6) i (7).
Izjednačavanjem dobivamo:

- Uz f:

  $$h = h(c_1 + c_2 + c_3 + c_4)$$

  odakle imamo

  $$c_1 + c_2 + c_3 + c_4 = 1 \tag{8}$$

- Uz f_x:

  $$\frac{h^2}{2} = h(c_2 h\alpha_2 + c_3 h\alpha_3 + c_4 h\alpha_4)$$

  odakle imamo

  $$c_2\alpha_2 + c_3\alpha_3 + c_4\alpha_4 = \frac{1}{2} \tag{9}$$

- Uz $f_y f$:

  $$\frac{h^2}{2} = h(c_2\beta_{21}h + c_3(\beta_{31}h + \beta_{32}h) + c_4(\beta_{41}h + \beta_{42}h + \beta_{43}h))$$

  odakle imamo

  $$c_2\beta_{21} + c_3(\beta_{31} + \beta_{32}) + c_4(\beta_{41} + \beta_{42} + \beta_{43}) = \frac{1}{2} \tag{10}$$

- Uz f_{xx}:

$$\frac{h^3}{6} = h(c_2\frac{h^2}{2}\alpha_2^2 + c_3\frac{h^2}{2}\alpha_3^2 + c_4\frac{h^2}{2}\alpha_4^2)$$

odakle imamo

$$c_2\alpha_2^2 + c_3\alpha_3^2 + c_4\alpha_4^2 = \frac{1}{3} \tag{11}$$

- Uz $f_{xy}f$:

$$2\frac{h^3}{6} = h(c_2\frac{h^2}{2}2\alpha_2\beta_{21} + c_3\frac{h^2}{2}(2\alpha_3\beta_{31} + 2\alpha_3\beta_{32}) + c_4\frac{h^2}{2}(2\alpha_4\beta_{41} + 2\alpha_4\beta_{42} + 2\alpha_4\beta_{43}))$$

odakle imamo

$$c_2\alpha_2\beta_{21} + c_3\alpha_3(\beta_{31} + \beta_{32}) + c_4\alpha_4(\beta_{41} + \beta_{42} + \beta_{43}) = \frac{1}{3} \tag{12}$$

- Uz $f_{yy}f^2$:

$$\begin{aligned}\frac{h^3}{6} = h(c_2\frac{h^2}{2}\beta_{21}^2 + c_3(\frac{h^2}{2}\beta_{31}^2 + \frac{h^2}{2}2\beta_{31}\beta_{32} + \frac{h^2}{2}\beta_{32}^2) + c_4(\frac{h^2}{2}\beta_{41}^2 + \frac{h^2}{2}\beta_{42}^2 + \frac{h^2}{2}\beta_{43}^2 \\ + 2\frac{h^2}{2}\beta_{41}\beta_{42} + 2\frac{h^2}{2}\beta_{41}\beta_{43} + 2\frac{h^2}{2}\beta_{42}\beta_{43}))\end{aligned}$$

odakle imamo

$$c_2\beta_{21}^2 + c_3(\beta_{31} + \beta_{32})^2 + c_4(\beta_{41} + \beta_{42} + \beta_{43})^2 = \frac{1}{3} \tag{13}$$

- Uz $f_x f_y$:

$$\frac{h^3}{6} = h(c_3 hh\beta_{32}\alpha_2 + c_4(h\beta_{42}h\alpha_2 + h\beta_{43}h\alpha_3))$$

odakle imamo

$$c_3\beta_{32}\alpha_2 + c_4(\beta_{42}\alpha_2 + \beta_{43}\alpha_3) = \frac{1}{6} \tag{14}$$

- Uz $f_y^2 f$:

$$\frac{h^3}{6} = h(c_3 h\beta_{32}h\beta_{21} + c_4(h\beta_{42}h\beta_{21} + h\beta_{43}h\beta_{31} + h\beta_{43}h\beta_{32}))$$

odakle imamo

$$c_3\beta_{32}\beta_{21} + c_4(\beta_{42}\beta_{21} + \beta_{43}\beta_{31} + \beta_{43}\beta_{32}) = \frac{1}{6} \tag{15}$$

- Uz f_{xxx}:

$$\frac{h^4}{24} = h(c_2\frac{h^3}{6}\alpha_2^3 + c_3\frac{h^3}{6}\alpha_3^3 + c_4\frac{h^3}{6}\alpha_4^3)$$

odakle imamo

$$c_2\alpha_2^3 + c_3\alpha_3^3 + c_4\alpha_4^3 = \frac{1}{4} \tag{16}$$

- Uz $f_{xxy}f$:

$$3\frac{h^4}{24} = h(c_2\frac{h^3}{6}3\alpha_2^2\beta_{21} + c_3(\frac{h^3}{6}3\alpha_3^2\beta_{31} + \frac{h^3}{6}3\alpha_3^2\beta_{32})$$
$$+ c_4(\frac{h^3}{6}3\alpha_4^2\beta_{41} + \frac{h^3}{6}3\alpha_4^2\beta_{42} + \frac{h^3}{6}3\alpha_4^2\beta_{43}))$$

odakle imamo

$$c_2\alpha_2^2\beta_{21} + c_3\alpha_3^2(\beta_{31}+\beta_{32}) + c_4\alpha_4^2(\beta_{41}+\beta_{42}+\beta_{43}) = \frac{1}{4} \tag{17}$$

- Uz $f_{xyy}f^2$:

$$3\frac{h^4}{24} = h(c_2\frac{h^3}{6}3\alpha_2\beta_{21}^2 + c_3(\frac{h^3}{6}3\alpha_3\beta_{31}^2 + \frac{h^3}{6}3\alpha_3\beta_{32}^2 + \frac{h^3}{6}3\alpha_3 2\beta_{31}\beta_{32})$$
$$+ c_4(\frac{h^3}{6}3\alpha_4\beta_{41}^2 + \frac{h^3}{6}3\alpha_4\beta_{42}^2 + \frac{h^3}{6}3\alpha_4\beta_{43}^2 + \frac{h^3}{6}3\alpha_4 2\beta_{41}\beta_{42}$$
$$+ \frac{h^3}{6}3\alpha_4 2\beta_{41}\beta_{43} + \frac{h^3}{6}3\alpha_4 2\beta_{42}\beta_{43}))$$

odakle imamo

$$c_2\alpha_2\beta_{21}^2 + c_3\alpha_3(\beta_{31}+\beta_{32})^2 + c_4\alpha_4(\beta_{41}+\beta_{42}+\beta_{43})^2 = \frac{1}{4} \tag{18}$$

- Uz $f_{xy}f_x$:

$$3\frac{h^4}{24} = h(c_3\frac{h^2}{2}2\alpha_3\beta_{32}h\alpha_2 + c_4(\frac{h^2}{2}2\alpha_4\beta_{42}h\alpha_2 + \frac{h^2}{2}2\alpha_4\beta_{43}h\alpha_3))$$

odakle imamo

$$c_3\alpha_2\alpha_3\beta_{32} + c_4\alpha_4(\alpha_2\beta_{42}+\alpha_3\beta_{43}) = \frac{1}{8} \tag{19}$$

- Uz $f_{xy}f_yf$:

$$5\frac{h^4}{24} = h(c_3(\frac{h^2}{2}2\alpha_3\beta_{32}h\beta_{21} + \beta_{32}h\alpha_2\beta_{21}h^2) + c_4(\frac{h^2}{2}2\alpha_2\beta_{21}h\beta_{42}$$
$$+ \frac{h^2}{2}2\alpha_4\beta_{43}h\beta_{31} + \frac{h^2}{2}2\alpha_4\beta_{43}h\beta_{32} + \frac{h^2}{2}2\alpha_3\beta_{43}h\beta_{31}$$
$$+ \frac{h^2}{2}2\alpha_3\beta_{43}h\beta_{32} + \frac{h^2}{2}2\alpha_4\beta_{21}h\beta_{42}))$$

odakle imamo

$$c_3\beta_{32}\beta_{21}(\alpha_2+\alpha_3) + c_4(\alpha_2\beta_{21}\beta_{42} + \alpha_4\beta_{21}\beta_{42} + \alpha_4\beta_{43}(\beta_{31}+\beta_{32}) + \alpha_3\beta_{43}(\beta_{31}+\beta_{32})) = \frac{5}{24} \tag{20}$$

- Uz $f_{yyy}f^3$:

$$\frac{h^4}{24} = h(c_2\frac{h^3}{6}\beta_{21}^3 + c_3(\frac{h^3}{6}\beta_{31}^3 + \frac{h^3}{6}\beta_{32}^3 + \frac{h^3}{6}3\beta_{31}^2\beta_{32} + \frac{h^3}{6}3\beta_{31}\beta_{32}^2) + c_4(\frac{h^3}{6}\beta_{41}^3 + \frac{h^3}{6}\beta_{42}^3 + \frac{h^3}{6}\beta_{43}^3 + \frac{h^3}{6}3\beta_{41}^2\beta_{42} + \frac{h^3}{6}3\beta_{41}\beta_{42}^2 + \frac{h^3}{6}3\beta_{41}^2\beta_{43} + \frac{h^3}{6}3\beta_{41}\beta_{43}^2 + \frac{h^3}{6}3\beta_{42}^2\beta_{43} + \frac{h^3}{6}3\beta_{42}\beta_{43}^2))$$

odakle imamo

$$c_2\beta_{21}^3 + c_3(\beta_{31}+\beta_{32})^3 + c_4(\beta_{41}+\beta_{42}+\beta_{43})^3 = \frac{1}{4} \tag{21}$$

- Uz $f_x f_{yy} f$:

$$3\frac{h^4}{24} = h(c_3(\frac{h^2}{2}\beta_{32}^2 2\alpha_2 h + \beta_{31}\beta_{32}h^2\alpha_2 h) + c_4(\frac{h^2}{2}\beta_{42}^2 2\alpha_2 h + \frac{h^2}{2}\beta_{43}^2 2\alpha_3 h + \frac{h^2}{2}2\beta_{41}\beta_{42}\alpha_2 h + \frac{h^2}{2}2\beta_{41}\beta_{43}\alpha_3 h + \frac{h^2}{2}2\beta_{42}\beta_{43}\alpha_2 h + \frac{h^2}{2}2\beta_{42}\beta_{43}\alpha_3 h))$$

odakle imamo

$$c_3\alpha_2\beta_{32}(\beta_{31}+\beta_{32}) + c_4(\alpha_2\beta_{42}(\beta_{41}+\beta_{42}+\beta_{43}) + \alpha_3\beta_{43}(\beta_{41}+\beta_{42}+\beta_{43})) = \frac{1}{8} \tag{22}$$

- Uz $f_y f_{yy} f^2$:

$$4\frac{h^4}{24} = h(c_3(h\frac{h^2}{2}\beta_{32}\beta_{21}^2 + h\frac{h^2}{2}2\beta_{32}^2\beta_{21} + h\frac{h^2}{2}2\beta_{31}\beta_{32}\beta_{21}) + c_4(h\frac{h^2}{2}\beta_{42}\beta_{21}^2 + h\frac{h^2}{2}\beta_{43}\beta_{31}^2 + h\frac{h^2}{2}\beta_{43}\beta_{32}^2 + h\frac{h^2}{2}2\beta_{43}\beta_{31}\beta_{32} + h\frac{h^2}{2}2\beta_{42}^2\beta_{21} + h\frac{h^2}{2}2\beta_{43}^2\beta_{31} + h\frac{h^2}{2}2\beta_{43}^2\beta_{32} + h\frac{h^2}{2}2\beta_{41}\beta_{42}\beta_{21} + h\frac{h^2}{2}2\beta_{41}\beta_{43}\beta_{31} + h\frac{h^2}{2}2\beta_{41}\beta_{43}\beta_{32} + h\frac{h^2}{2}2\beta_{42}\beta_{43}\beta_{21} + h\frac{h^2}{2}2\beta_{42}\beta_{43}\beta_{31} + h\frac{h^2}{2}2\beta_{42}\beta_{43}\beta_{32}))$$

odakle imamo

$$c_3(\frac{1}{2}\beta_{32}\beta_{21}^2 + \frac{1}{2}\beta_{32}^2 2\beta_{21} + \beta_{31}\beta_{32}\beta_{21}) + c_4(\frac{1}{2}\beta_{42}\beta_{21}^2 + \beta_{43}(\frac{1}{2}\beta_{31}^2 + \frac{1}{2}\beta_{32}^2 + \beta_{31}\beta 32) + \beta_{42}^2\beta_{21} + \beta_{43}^2\beta_{31} + \beta_{43}^2\beta_{32} + \beta_{41}\beta_{42}\beta_{21} + \beta_{41}\beta_{43}(\beta_{31}+\beta_{32}) + \beta_{42}\beta_{43}(\beta_{21}+\beta_{31}+\beta_{32})) = \frac{1}{6} \tag{23}$$

- Uz $f_{xx}f_y$:

$$\frac{h^4}{24} = h(c_3 h\frac{h^2}{2}\beta_{32}\alpha_2^2 + c_4(h\frac{h^2}{2}\beta_{42}\alpha_2^2 + h\frac{h^2}{2}\beta_{43}\alpha_3^2))$$

odakle imamo

$$c_3\alpha_2^2\beta_{32} + c_4(\alpha_2^2\beta_{42} + \alpha_3^2\beta_{43}) = \frac{1}{12} \tag{24}$$

- Uz $f_x f_y^2$:

$$\frac{h^4}{24} = h(c_4 h\beta_{43} h\beta_{32} h\alpha_2)$$

odakle imamo

$$c_4\alpha_2\beta_{32}\beta_{43} = \frac{1}{24} \tag{25}$$

- Uz $f_y^3 f$:

$$\frac{h^4}{24} = h(c_4 h\beta_{43} h\beta_{32} h\beta_{21})$$

odakle imamo

$$c_4\beta_{21}\beta_{32}\beta_{43} = \frac{1}{24} \tag{26}$$

Sada izjednačavanjem (1) i (1) imamo $\boxed{\alpha_2 = \beta_{21}}$
Izjednačavanjem (14) i (15) imamo
$c_3\beta_{32}\alpha_2 + c_4(\beta_{42}\alpha_2 + \beta_{43}\alpha_3) = c_3\beta_{32}\beta_{21} + c_4(\beta_{42}\beta_{21} + \beta_{43}\beta_{31} + \beta_{43}\beta_{32})$
$\beta_{43}\alpha_3 = \beta_{43}\beta_{31} + \beta_{43}\beta_{32}$
$\boxed{\alpha_3 = \beta_{31} + \beta_{32}}$
Izjednačavanjem (16) i (17) imamo
$c_2\alpha_2^3 + c_3\alpha_3^3 + c_4\alpha_4^3 = c_2\alpha_2^2\beta_{21} + c_3\alpha_3^2(\beta_{31} + \beta_{32}) + c_4\alpha_4^2(\beta_{41} + \beta_{42} + \beta_{43})$
$\boxed{\alpha_4 = \beta_{41} + \beta_{42} + \beta_{43}}$
Iz (9) i (10) imamo $\boxed{c_2\alpha_2 + c_3\alpha_3 + c_4\alpha_4 = \frac{1}{2}}$
Iz (11), (12) i (13) imamo $\boxed{c_2\alpha_2^2 + c_3\alpha_3^2 + c_4\alpha_4^2 = \frac{1}{3}}$
Iz (18) i (21) imamo $\boxed{c_2\alpha_2^3 + c_3\alpha_3^3 + c_4\alpha_4^3 = \frac{1}{4}}$
Iz (8) imamo $\boxed{c_1 + c_2 + c_3 + c_4 = 1}$
Iz (14) i (15) imamo $\boxed{c_3\alpha_2\beta_{32} + c_4(\alpha_2\beta_{42} + \alpha_3\beta_{43}) = \frac{1}{6}}$
Iz (19) i (22) imamo $\boxed{c_3\alpha_2\alpha_3\beta_{32} + c_4\alpha_4(\alpha_2\beta_{42} + \alpha_3\beta_{43}) = \frac{1}{8}}$
Iz (24) imamo $\boxed{c_3\alpha_2^2\beta_{32} + c_4(\alpha_2^2\beta_{42} + \alpha_3^2\beta_{43}) = \frac{1}{12}}$ (!)
Iz (25) i (26) imamo $\boxed{c_4\alpha_2\beta_{32}\beta_{43} = \frac{1}{24}}$
Iz (20) imamo
$c_3\beta_{32}\beta_{21}(\alpha_2 + \alpha_3) + c_4(\alpha_2\beta_{21}\beta_{42} + \alpha_4\beta_{21}\beta_{42} + \alpha_4\beta_{43}(\beta_{31} + \beta_{32}) + \alpha_3\beta_{43}(\beta_{31} + \beta_{32})) = \frac{5}{24}$
$c_3\beta_{32}\alpha_2^2 + c_3\beta_{32}\alpha_2\alpha_3 + c_4\beta_{42}\alpha_2^2 + c_4\alpha_4\alpha_2\beta_{42} + c_4\alpha_4\alpha_3\beta_{43} + c_4\alpha_3^2\beta_{43} = \frac{5}{24}$
$\underbrace{c_3\alpha_2^2\beta_{32} + c_4(\alpha_2^2\beta_{42} + \alpha_3^2\beta_{43})}_{\text{prema (!)} = \frac{1}{12}} + c_3\beta_{32}\alpha_2\alpha_3 + c_4\alpha_2\alpha_4\beta_{42} + c_4\alpha_3\alpha_4\beta_{43} = \frac{5}{24}$

$c_3\alpha_2\alpha_3\beta_{32} + c_4\alpha_4(\alpha_2\beta_{42} + \alpha_3\beta_{43}) = \frac{1}{8}$ (to smo već dobili iz (19) i (22))
Iz (23) imamo
$c_3(\frac{1}{2}\beta_{32}\beta_{21}^2 + \frac{1}{2}\beta_{32}^2 2\beta_{21} + \beta_{31}\beta_{32}\beta_{21}) + c_4(\frac{1}{2}\beta_{42}\beta_{21}^2 + \beta_{43}(\frac{1}{2}\beta_{31}^2 + \frac{1}{2}\beta_{32}^2 + \beta_{31}\beta_{32}) + \beta_{42}^2\beta_{21} + \beta_{43}^2\beta_{31} + \beta_{43}^2\beta_{32} + \beta_{41}\beta_{42}\beta_{21} + \beta_{41}\beta_{43}(\beta_{31} + \beta_{32}) + \beta_{42}\beta_{43}(\beta_{21} + \beta_{31} + \beta_{32})) = \frac{1}{6}$
$\frac{1}{2}c_3\alpha_2^2\beta_{32} + \frac{1}{2}c_4(\alpha_2^2\beta_{42} + \alpha_3^2\beta_{43}) + c_3\beta_{21}\beta_{32}(\beta_{31} + \beta_{32}) + c_4\beta_{21}\beta_{42}(\beta_{41} + \beta_{42} + \beta_{43}) + c_4\alpha_3\beta_{43}(\beta_{41} + \beta_{42} + \beta_{43}) = \frac{1}{6}$
$c_3\beta_{21}\beta_{32}(\beta_{31} + \beta_{32}) + c_4\beta_{21}\beta_{42}(\beta_{41} + \beta_{42} + \beta_{43}) + c_4\alpha_3\beta_{43}(\beta_{41} + \beta_{42} + \beta_{43}) = \frac{1}{8}$ (to smo već dobili iz (19))
Sada imamo sustav od 11 jednadžbi s 13 nepoznanica:
$\beta_{21} = \alpha_2$
$\beta_{31} + \beta_{32} = \alpha_3$
$\beta_{41} + \beta_{42} + \beta_{43} = \alpha_4$
$c_2\alpha_2 + c_3\alpha_3 + c_4\alpha_4 = \frac{1}{2}$
$c_2\alpha_2^2 + c_3\alpha_3^2 + c_4\alpha_4^2 = \frac{1}{3}$
$c_2\alpha_2^3 + c_3\alpha_3^3 + c_4\alpha_4^3 = \frac{1}{4}$
$c_1 + c_2 + c_3 + c_4 = 1$
$c_3\alpha_2\beta_{32} + c_4(\alpha_2\beta_{42} + \alpha_3\beta_{43}) = \frac{1}{6}$
$c_3\alpha_2\alpha_3\beta_{32} + c_4\alpha_4(\alpha_2\beta_{42} + \alpha_3\beta_{43}) = \frac{1}{8}$
$c_3\alpha_2^2\beta_{32} + c_4(\alpha_2^2\beta_{42} + \alpha_3^2\beta_{43}) = \frac{1}{12}$
$c_4\alpha_2\beta_{32}\beta_{43} = \frac{1}{24}$

Obzirom da je broj nepoznanica veći od broja jednadžbi, dvije nepoznanice odabiremo proizvoljno.
Uzmimo $\beta_{31} = 0$, $\alpha_2 = \frac{1}{2}$. Sada imamo

$$\boxed{\beta_{21} = \frac{1}{2}}$$
$$\beta_{32} = \alpha_3$$
$$\beta_{41} + \beta_{42} + \beta_{43} = \alpha_4$$
$$\tfrac{1}{2}c_2 + c_3\alpha_3 + c_4\alpha_4 = \tfrac{1}{2}$$
$$\tfrac{1}{4}c_2 + c_3\alpha_3^2 + c_4\alpha_4^2 = \tfrac{1}{3}$$
$$\tfrac{1}{8}c_2 + c_3\alpha_3^3 + c_4\alpha_4^3 = \tfrac{1}{4}$$
$$c_1 + c_2 + c_3 + c_4 = 1$$
$$\tfrac{1}{2}c_3\alpha_3 + c_4(\tfrac{1}{2}\beta_{42} + \alpha_3\beta_{43}) = \tfrac{1}{6}$$
$$\tfrac{1}{2}c_3\alpha_3^2 + c_4\alpha_4(\tfrac{1}{2}\beta_{42} + \alpha_3\beta_{43}) = \tfrac{1}{8}$$
$$\tfrac{1}{4}c_3\alpha_3 + c_4(\tfrac{1}{4}\beta_{42} + \alpha_3^2\beta_{43}) = \tfrac{1}{12}$$
$$\tfrac{1}{2}c_4\alpha_3\beta_{43} = \tfrac{1}{24}$$

$$\tfrac{1}{2}c_3\alpha_3 + \tfrac{1}{2}c_4\beta_{42} + c_4\alpha_3\beta_{43} = \tfrac{1}{6}$$
$$\tfrac{1}{2}c_3\alpha_3^2 + \tfrac{1}{2}c_4\alpha_4\beta_{42} + c_4\alpha_4\alpha_3\beta_{43} = \tfrac{1}{8}$$
$$\tfrac{1}{4}c_3\alpha_3 + \tfrac{1}{4}c_4\beta_{42} + c_4\alpha_3^2\beta_{43} = \tfrac{1}{12}$$
$$\tfrac{1}{2}c_4\alpha_3\beta_{43} = \tfrac{1}{24} \quad\Rightarrow\quad c_4\alpha_3\beta_{43} = \tfrac{1}{12}$$

$$\tfrac{1}{2}c_3\alpha_3 + \tfrac{1}{2}c_4\beta_{42} = \tfrac{1}{12} / \cdot (-1)$$
$$\tfrac{1}{2}c_3\alpha_3^2 + \tfrac{1}{2}c_4\alpha_4\beta_{42} + \tfrac{1}{12}\alpha_4 = \tfrac{1}{8}$$
$$\tfrac{1}{4}c_3\alpha_3 + \tfrac{1}{4}c_4\beta_{42} + \tfrac{1}{12}\alpha_3 = \tfrac{1}{12} / \cdot 2$$

$$-\tfrac{1}{2}c_3\alpha_3 - \tfrac{1}{2}c_4\beta_{42} = -\tfrac{1}{12}$$
$$\tfrac{1}{2}c_3\alpha_3^2 + \tfrac{1}{2}c_4\alpha_4\beta_{42} + \tfrac{1}{12}\alpha_4 = \tfrac{1}{8}$$
$$\tfrac{1}{2}c_3\alpha_3 + \tfrac{1}{2}c_4\beta_{42} + \tfrac{1}{6}\alpha_3 = \tfrac{1}{6}$$

Zbrojimo 1. i 3. jednadžbu i dobivamo

$$\tfrac{1}{6}\alpha_3 = \tfrac{1}{12}$$
$$\boxed{\alpha_3 = \frac{1}{2} \quad\Rightarrow\quad \beta_{32} = \frac{1}{2}}$$

Sada dobivamo sustav:

$\beta_{41} + \beta_{42} + \beta_{43} = \alpha_4$
$\frac{1}{2}c_2 + \frac{1}{2}c_3 + c_4\alpha_4 = \frac{1}{2}$
$\frac{1}{4}c_2 + \frac{1}{4}c_3 + c_4\alpha_4^2 = \frac{1}{3}$
$\frac{1}{8}c_2 + \frac{1}{8}c_3 + c_4\alpha_4^3 = \frac{1}{4}$
$c_1 + c_2 + c_3 + c_4 = 1$
$\frac{1}{8}c_3 + \frac{1}{2}c_4\alpha_4\beta_{42} + \frac{1}{2}c_4\alpha_4\beta_{43} = \frac{1}{8}$
$\frac{1}{8}c_3 + \frac{1}{4}c_4\beta_{42} + \frac{1}{4}c_4\beta_{43} = \frac{1}{12}$
$c_4\beta_{43} = \frac{1}{6}$

$\beta_{41} + \beta_{42} + \beta_{43} = \alpha_4$
$c_2 + c_3 + 2c_4\alpha_4 = 1$

$c_2 + c_3 + 4c_4\alpha_4^2 = \frac{4}{3}$
$c_2 + c_3 + 8c_4\alpha_4^3 = 2$
$c_1 + c_2 + c_3 + c_4 = 1$
$c_3 + 4c_4\alpha_4\beta_{42} + \frac{2}{3}\alpha_4 = 1$
$c_3 + 2c_4\beta_{42} = \frac{1}{3}$
$c_4\beta_{43} = \frac{1}{6}$
Pomoću Gaussovih eliminacija rješavamo sustav

$$\begin{cases} c_2 + c_3 + 2c_4\alpha_4 = 1 \\ c_2 + c_3 + 4c_4\alpha_4^2 = \frac{4}{3} \\ c_2 + c_3 + 8c_4\alpha_4^3 = 2 \end{cases}$$

po nepoznanicama c_2, c_3, c_4:

$$\begin{bmatrix} 1 & 1 & 2\alpha_4 & | & 1 \\ 1 & 1 & 4\alpha_4^2 & | & \frac{4}{3} \\ 1 & 1 & 8\alpha_4^3 & | & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 2\alpha_4 & | & 1 \\ 0 & 0 & 4\alpha_4^2 - 2\alpha_4 & | & \frac{1}{3} \\ 0 & 0 & 8\alpha_4^3 - 2\alpha_4 & | & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 2\alpha_4 & | & 1 \\ 0 & 0 & 4\alpha_4^2 - 2\alpha_4 & | & \frac{1}{3} \\ 0 & 0 & 0 & | & -\frac{2}{3}\alpha_4 + \frac{2}{3} \end{bmatrix}$$

Sada slijedi:
$-\frac{2}{3}\alpha_4 + \frac{2}{3} = 0$
$\boxed{\alpha_4 = 1}$
$(4\alpha_4^2 - 2\alpha_4)c_4 = \frac{1}{3}$
$2c_4 = \frac{1}{3}$
$\boxed{c_4 = \frac{1}{6}.}$
Sada je naš sustav oblika:

$$\begin{gathered} \beta_{41} + \beta_{42} + \beta_{43} = 1 \\ c_1 + c_2 + c_3 = \frac{5}{6} \\ c_2 + c_3 + \frac{1}{3} = 1 \\ c_3 + 4\frac{1}{6}\beta_{42} + \frac{2}{3} = 1 \\ c_3 + 2\frac{1}{6}\beta_{42} = \frac{1}{3} \\ \frac{1}{6}\beta_{43} = \frac{1}{6} \end{gathered}$$

$$\begin{gathered} \beta_{41} + \beta_{42} + \beta_{43} = 1 \\ c_1 + c_2 + c_3 = \frac{5}{6} \\ c_2 + c_3 = \frac{2}{3} \\ c_3 + \frac{2}{3}\beta_{42} = \frac{1}{3} \\ c_3 + \frac{1}{3}\beta_{42} = \frac{1}{3} \\ \boxed{\beta_{43} = 1} \end{gathered}$$

Riješimo sada sustav:

$$\begin{aligned} c_1 + c_2 + c_3 &= \frac{5}{6} \\ c_2 + c_3 &= \frac{2}{3} \end{aligned}$$

$$-c_1 = -\tfrac{1}{6}$$
$$\boxed{c_1 = \frac{1}{6}}$$
$$c_2 + c_3 = \tfrac{2}{3}$$
$$\boxed{c_2 = \frac{1}{3}}$$

Riješimo i sustav

$$c_3 + \frac{2}{3}\beta_{42} = \frac{1}{3}$$
$$c_3 + \frac{1}{3}\beta_{42} = \frac{1}{3}$$

$$-\tfrac{1}{3}\beta_{42} = 0$$
$$\boxed{\beta_{42} = 0}$$
$$\boxed{c_3 = \frac{1}{3}}$$

Sada imamo
$\beta_{41} + \beta_{42} + \beta_{43} = 1$
$\boxed{\beta_{41} = 0}$
Rješavajući Runge Kutta sustav dobili smo parametre
$c_1 = \frac{1}{6},\ \ c_2 = \frac{1}{3},\ \ c_3 = \frac{1}{3},\ \ c_4 = \frac{1}{6}$
$\alpha_2 = \frac{1}{2},\ \ \alpha_3 = \frac{1}{2},\ \ \alpha_4 = 1$
$\beta_{21} = \frac{1}{2},\ \ \beta_{31} = 0,\ \ \beta_{32} = \frac{1}{2},\ \ \beta_{41} = 0,\ \ \beta_{42} = 0,\ \ \beta_{43} = 1$
Runge Kutta metoda četvrtog reda je oblika

$$y_{k+1} = y_k + h(c_1 k_1^{(k)} + c_2 k_2^{(k)} + c_3 k_3^{(k)} + c_4 k_4^{(k)})$$
$$y_{k+1} = y_k + h(\tfrac{1}{6} k_1^{(k)} + \tfrac{1}{3} k_2^{(k)} + \tfrac{1}{3} k_3^{(k)} + \tfrac{1}{6} k_4^{(k)})$$
$$y_{k+1} = y_k + \tfrac{h}{6}(k_1^{(k)} + 2k_2^{(k)} + 2k_3^{(k)} + k_4^{(k)})$$
$$k_1^{(k)} = f(x_k, y_k)$$
$$k_2^{(k)} = f(x_k + \tfrac{h}{2}, y_k + \tfrac{h}{2} k_1^{(k)})$$
$$k_3^{(k)} = f(x_k + \tfrac{h}{2}, y_k + \tfrac{h}{2} k_2^{(k)})$$
$$k_4^{(k)} = f(x_k + h, y_k + h k_3^{(k)})$$

3.1.1 Primjeri Runge Kutta metode četvrtoga reda

Primjer 3.1.

Runge Kutta metodom četvrtoga reda riješite diferencijalnu jednadžbu $y' = \frac{x^2 - y}{x}$ *s početnim uvjetom* $y(1) = 1$ *na* $[1, 2.2]$ *uz korak* 0.3.

Rješenje:
Elementi subdivizije segmenta $[1, 2.2]$ glase $x_i = x_0 + ih = 0.5i$, $i = 0, \ldots, 4$. Potrebno je odrediti vrijednosti y_{k+1} za $k = 0, \ldots, 3$. Rezultate ćemo zapisati u tablicu.

Zapišimo najprije opći oblik Runge Kutta metode četvrtoga reda:

$$\begin{gathered}
y_{k+1} = y_k + \frac{h}{6}(k_1^{(k)} + 2k_2^{(k)} + 2k_3^{(k)} + k_4^{(k)}) \\
k_1^{(k)} = f(x_k, y_k) \\
k_2^{(k)} = f(x_k + \frac{h}{2}, y_k + \frac{h}{2}k_1^{(k)}) \\
k_3^{(k)} = f(x_k + \frac{h}{2}, y_k + \frac{h}{2}k_2^{(k)}) \\
k_4^{(k)} = f(x_k + h, y_k + hk_3^{(k)})
\end{gathered}$$

Odredimo sada y_1.

$$\begin{gathered}
y_1 = y_0 + \frac{h}{6}(k_1^{(0)} + 2k_2^{(0)} + 2k_3^{(0)} + k_4^{(0)}) \\
k_1^{(0)} = f(x_0, y_0) = f(1,1) = 0 \\
k_2^{(0)} = f(1 + \frac{0.3}{2}, 1 + \frac{0.3}{2}k_1^{(0)}) = f(1.15, 1) = 0.2804 \\
k_3^{(0)} = f(1 + \frac{0.3}{2}, 1 + \frac{0.3}{2}k_2^{(0)}) = f(1.15, 1.0421) = 0.2438 \\
k_4^{(0)} = f(1 + 0.3, 1 + 0.3k_3^{(0)}) = f(1.3, 1.0731) = 0.4745
\end{gathered}$$

Vratimo $k_1^{(0)}, k_2^{(0)}, k_3^{(0)} i k_4^{(0)}$ u y_1 i dobivamo

$$y_1 = 1 + \frac{0.3}{6}(0 + 2 \cdot 0.2804 + 2 \cdot 0.2438 + 0.4745) = 1.0761$$

Ponovimo postupak kako bi pronašli preostale y_{k+1}.

$$\begin{gathered}
y_2 = y_1 + \frac{h}{6}(k_1^{(1)} + 2k_2^{(1)} + 2k_3^{(1)} + k_4^{(1)}) \\
k_1^{(1)} = f(x_1, y_1) = f(1.3, 1.0761) = 0.4722 \\
k_2^{(1)} = f(x_1 + \frac{h}{2}, y_1 + \frac{h}{2}k_1^{(1)}) = f(1.45, 1.1469) = 0.659 \\
k_3^{(1)} = f(x_1 + \frac{h}{2}, y_1 + \frac{h}{2}k_2^{(1)}) = f(1.45, 1.175) = 0.6397 \\
k_4^{(1)} = f(x_1 + h, y_1 + hk_3^{(1)}) = f(1.6, 1.268) = 0.8075 \\
y_2 = 1.27
\end{gathered}$$

$$\begin{gathered}
y_3 = y_2 + \frac{h}{6}(k_1^{(2)} + 2k_2^{(2)} + 2k_3^{(2)} + k_4^{(2)}) \\
k_1^{(2)} = f(x_2, y_2) = f(1.6, 1.27) = 0.8063 \\
k_2^{(2)} = f(x_2 + \frac{h}{2}, y_2 + \frac{h}{2}k_1^{(2)}) = f(1.75, 1.3909) = 0.9552 \\
k_3^{(2)} = f(x_2 + \frac{h}{2}, y_2 + \frac{h}{2}k_2^{(2)}) = f(1.75, 1.4133) = 0.9424 \\
k_4^{(2)} = f(x_2 + h, y_2 + hk_3^{(2)}) = f(1.9, 1.5527) = 1.0828 \\
y_3 = 1.5542
\end{gathered}$$

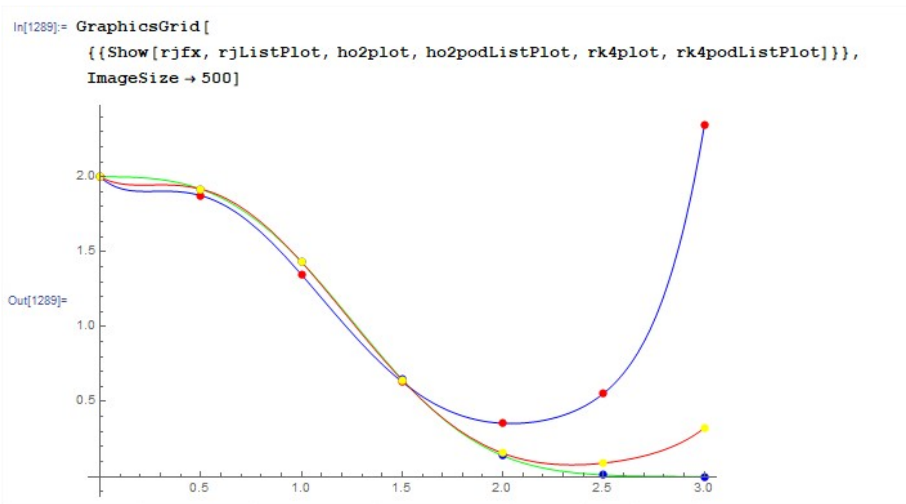
$$y_4 = y_3 + \frac{h}{6}(k_1^{(3)} + 2k_2^{(3)} + 2k_3^{(3)} + k_4^{(3)})$$
$$k_1^{(3)} = f(x_3, y_3) = f(1.9, 1.5542) = 1.082$$
$$k_2^{(3)} = f(x_3 + \frac{h}{2}, y_3 + \frac{h}{2}k_1^{(3)}) = f(2.05, 1.7165) = 1.2127$$
$$k_3^{(3)} = f(x_3 + \frac{h}{2}, y_3 + \frac{h}{2}k_2^{(3)}) = f(2.05, 1.7361) = 1.2031$$
$$k_4^{(3)} = f(x_3 + h, y_3 + hk_3^{(3)}) = f(2.2, 1.9151) = 1.3295$$
$$y_4 = 1.9164$$

k	x_k	y_k
0	1	1
1	1.3	1.0761
2	1.6	1.27
3	1.9	1.5542
4	2.2	**1.9164**

Tablica 2: Cauchyjev problem $y' = \frac{x^2-y}{x}$, y(1)=1, $x \in [1, 2.2]$

Primjer 3.2.

Promotrimo diferencijalnu jednadžbu $y' = -x^2 y$ s početnim uvjetom $y(0) = 2$ na $[0,3]$ uz korak 0.5. Cilj ovoga primjera je grafički pokazati odstupanja dobivena rješavanjem dane jednadžbe egzaktno, RK-2 te RK-4 metodom. Rješenja RK-2 i RK-4 metode prikazat ćemo u tablici.
Na grafu je egzaktno rješenje diferencijalne jednadžbe prikazano zelenom krivuljom. Točke dobivene rješavanjem polazne diferencijalne jednadžbe RK-2 metodom prikazane su crvenom bojom, a pripadni interpolacijski polinom određen tim točkama dan je plavom bojom. Kao rješenje diferencijalne jednadžbe RK-4 metodom također dobivamo točke i one su označene žuto, a interpolacijski polinom kroz njih crvenom bojom.



Slika 1. Rješenje diferencijalne jednadžbe $y' = -x^2 y$

k	x_k	$y_k(RK-2)$	$y_k(RK-4)$
0	0	2	2
1	0.5	1.875	1.91827
2	1	1.34766	1.43276
3	1.5	0.63171	0.64947
4	2	0.3553	0.16617
5	2.5	0.55522	0.1031
6	3	2.34232	0.38036

Tablica 3: Cauchyjev problem $y' = -x^2 y$, y(0)=2, $x \in [0,3]$

3.2 Rješavanje sustava diferencijalnih jednadžbi Runge Kutta metodom

Pomoću Runge Kutta metode možemo riješiti i sustave diferencijalnih jednadžbi (vidi [1] i [3]).
Za sustav:
$y' = f(x,y,z)$
$z' = g(x,y,z)$
$y(x_0) = y_0,\ \ z(x_0) = z_0$
imamo:

$$y_{k+1} = y_k + \frac{h}{6}(k_1^{(k)} + 2k_2^{(k)} + 2k_3^{(k)} + k_4^{(k)})$$
$$z_{k+1} = z_k + \frac{h}{6}(m_1^{(k)} + 2m_2^{(k)} + 2m_3^{(k)} + m_4^{(k)})$$

gdje su:

$$\begin{aligned} k_1^{(k)} &= f(x_k, y_k, z_k) \\ k_2^{(k)} &= f(x_k + \tfrac{h}{2}, y_k + \tfrac{h}{2}k_1^{(k)}, z_k + \tfrac{m_1^{(k)}}{2}) \\ k_3^{(k)} &= f(x_k + \tfrac{h}{2}, y_k + \tfrac{h}{2}k_2^{(k)}, z_k + \tfrac{m_2^{(k)}}{2}) \\ k_4^{(k)} &= f(x_k + h, y_k + hk_3^{(k)}, z_k + m_3^{(k)}) \end{aligned}$$

$$\begin{aligned} m_1^{(k)} &= g(x_k, y_k, z_k) \\ m_2^{(k)} &= g(x_k + \tfrac{h}{2}, y_k + \tfrac{h}{2}k_1^{(k)}, z_k + \tfrac{m_1^{(k)}}{2}) \\ m_3^{(k)} &= g(x_k + \tfrac{h}{2}, y_k + \tfrac{h}{2}k_2^{(k)}, z_k + \tfrac{m_2^{(k)}}{2}) \\ m_4^{(k)} &= g(x_k + h, y_k + hk_3^{(k)}, z_k + m_3^{(k)}) \end{aligned}$$

Diferencijalnu jednadžbu višeg reda možemo riješiti svođenjem na sustav diferencijalnih jednadžbi 1. reda (vidi [3]). Ako je dan Cauchyjev problem:
$y'' = g(x,y,y'),\ y(x_0) = \alpha,\ y'(x_0) = \beta,$
svodimo ga na sustav:
$y' = z,\ y(x_0) = y_0$
$z' = g(x,y,z),\ z(x_0) = z_0$
Taj sustav rješavamo Runge Kutta metodom:

$$y_1 = y_0 + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$
$$z_1 = z_0 + \frac{h}{6}(m_1 + 2m_2 + 2m_3 + m_4)$$

$$k_1 = hz_0$$
$$k_2 = h(z_0 + \frac{m_1}{2})$$
$$k_3 = h(z_0 + \frac{m_2}{2})$$
$$k_4 = h(z_0 + m_3)$$

$$m_1 = hg(x_0, y_0, z_0)$$
$$m_2 = hg(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{m_1}{2})$$
$$m_3 = hg(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{m_2}{2})$$
$$m_4 = hg(x_0 + h, y_0 + k_3, z_0 + m_3)$$

Primjer 3.3.

Pomoću RK-4 metode riješite diferencijalnu jednadžbu $y'' + 2y' + 3x = 5$ *s početnim uvjetima* $y(0) = 1$, $y'(0) = 2$ *na* $[0, 0.6]$ *uz korak* $h = 0.2$.

Rješenje:
Znamo $x_0 = 0$, $y_0 = 1$, $z_0 = 2$. Svodimo jednadžbu na sustav diferencijalnih jednadžbi 1. reda:
$y' = z$, $y(0) = 1$
$y'' = z' = 5 - 3x - 2z$, $y'(0) = 2$
Riješimo sustav.

$$y_1 = y_0 + \frac{h}{6}(k_1^{(0)} + 2k_2^{(0)} + 2k_3^{(0)} + k_4^{(0)})$$
$$k_1^{(0)} = f(x_0, y_0, z_0) = f(0, 1, 2) = 2$$
$$k_2^{(0)} = f(x_0 + \frac{h}{2}, y_0 + \frac{h}{2}k_1^{(0)}, z_0 + h\frac{m_1^{(0)}}{2}) = f(0.1, 1.2, 2.1) = 2.1$$
$$k_3^{(0)} = f(x_0 + \frac{h}{2}, y_0 + \frac{h}{2}k_2^{(0)}, z_0 + h\frac{m_2^{(0)}}{2}) = f(0.1, 1.21, 2.05) = 2.05$$
$$k_4^{(0)} = f(x_0 + h, y_0 + hk_3^{(0)}, z_0 + hm_3^{(0)}) = f(0.2, 1.41, 2.12) = 2.12$$
$$y_1 = 1.414$$

$$z_1 = z_0 + \frac{h}{6}(m_1^{(0)} + 2m_2^{(0)} + 2m_3^{(0)} + m_4^{(0)})$$
$$m_1^{(0)} = g(x_0, y_0, z_0) = g(0, 1, 2) = 1$$
$$m_2^{(0)} = g(x_0 + \frac{h}{2}, y_0 + \frac{h}{2}k_1^{(0)}, z_0 + h\frac{m_1^{(0)}}{2}) = g(0.1, 1.2, 2.1) = 0.5$$
$$m_3^{(0)} = g(x_0 + \frac{h}{2}, y_0 + \frac{h}{2}k_2^{(0)}, z_0 + h\frac{m_2^{(0)}}{2}) = g(0.1, 1.21, 2.05) = 0.6$$
$$m_4^{(0)} = g(x_0 + h, y_0 + hk_3^{(0)}, z_0 + hm_3^{(0)}) = g(0.2, 1.41, 2.12) = 0.16$$
$$z_1 = 2.112$$

$$y_2 = y_1 + \frac{h}{6}(k_1^{(1)} + 2k_2^{(1)} + 2k_3^{(1)} + k_4^{(1)})$$
$$k_1^{(1)} = f(x_1, y_1, z_1) = f(0.2, 1.414, 2.112) = 2.112$$
$$k_2^{(1)} = f(x_1 + \frac{h}{2}, y_1 + \frac{h}{2}k_1^{(1)}, z_1 + h\frac{m_1^{(1)}}{2}) = f(0.3, 1.6252, 2.1296) = 2.1296$$
$$k_3^{(1)} = f(x_1 + \frac{h}{2}, y_1 + \frac{h}{2}k_2^{(1)}, z_1 + h\frac{m_2^{(1)}}{2}) = f(0.3, 1.627, 2.0961) = 2.0961$$
$$k_4^{(1)} = f(x_1 + h, y_1 + hk_3^{(1)}, z_1 + hm_3^{(1)}) = f(0.4, 1.8332, 2.0936) = 2.0936$$
$$y_2 = 1.8359$$

$$z_2 = z_1 + \frac{h}{6}(m_1^{(1)} + 2m_2^{(1)} + 2m_3^{(1)} + m_4^{(1)})$$
$$m_1^{(1)} = g(x_1, y_1, z_1) = g(0.2, 1.414, 2.112) = 0.176$$
$$m_2^{(1)} = g(x_1 + \frac{h}{2}, y_1 + \frac{h}{2}k_1^{(1)}, z_1 + h\frac{m_1^{(1)}}{2}) = g(0.3, 1.6252, 2.1296) = -0.1592$$
$$m_3^{(1)} = g(x_1 + \frac{h}{2}, y_1 + \frac{h}{2}k_2^{(1)}, z_1 + h\frac{m_2^{(1)}}{2}) = g(0.3, 1.627, 2.0961) = -0.0922$$
$$m_4^{(1)} = g(x_1 + h, y_1 + hk_3^{(1)}, z_1 + hm_3^{(1)}) = g(0.4, 1.8332, 2.0936) = -0.3872$$
$$z_2 = 2.0966$$

$$y_3 = y_2 + \frac{h}{6}(k_1^{(2)} + 2k_2^{(2)} + 2k_3^{(2)} + k_4^{(2)})$$
$$k_1^{(2)} = f(x_2, y_2, z_2) = f(0.4, 1.8359, 2.0966) = 2.0966$$
$$k_2^{(2)} = f(x_2 + \frac{h}{2}, y_2 + \frac{h}{2}k_1^{(2)}, z_2 + h\frac{m_1^{(2)}}{2}) = f(0.5, 2.0456, 2.0573) = 2.0573$$
$$k_3^{(2)} = f(x_2 + \frac{h}{2}, y_2 + \frac{h}{2}k_2^{(2)}, z_2 + h\frac{m_2^{(2)}}{2}) = f(0.5, 2.0416, 2.0351) = 2.0351$$
$$k_4^{(2)} = f(x_2 + h, y_2 + hk_3^{(2)}, z_2 + hm_3^{(2)}) = f(0.6, 2.2429, 1.9826) = 1.9826$$
$$y_3 = 2.2447$$

$$z_3 = z_2 + \frac{h}{6}(m_1^{(2)} + 2m_2^{(2)} + 2m_3^{(2)} + m_4^{(2)})$$
$$m_1^{(2)} = g(x_2, y_2, z_2) = g(0.4, 1.8359, 2.0966) = -0.3932$$
$$m_2^{(2)} = g(x_2 + \frac{h}{2}, y_2 + \frac{h}{2}k_1^{(2)}, z_2 + h\frac{m_1^{(2)}}{2}) = g(0.5, 2.0456, 2.0573) = -0.6146$$
$$m_3^{(2)} = g(x_2 + \frac{h}{2}, y_2 + \frac{h}{2}k_2^{(2)}, z_2 + h\frac{m_2^{(2)}}{2}) = g(0.5, 2.0416, 2.0351) = -0.5702$$
$$m_4^{(2)} = g(x_2 + h, y_2 + hk_3^{(2)}, z_2 + hm_3^{(2)}) = g(0.6, 2.2429, 1.9826) = -0.7652$$
$$z_3 = 2.0185$$

k	x_k	y_k	z_k
0	0	1	2
1	0.2	1.414	2.112
2	0.4	1.8359	2.0966
3	0.6	2.2447	2.0185

Tablica 4: Cauchyjev problem $y'' + 2y' + 3x = 5$, y(0)=1, y'(0)=2 $x \in [0, 0.6]$

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