SupOV processes

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SupOU processes

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1 | Introduction

Superpositions of Ornstein-Uhlenbeck type processes, known as supOU processes, belong to the class of stationary processes in continuous time. As the name suggests, they are obtained as the sums of independent Ornstein-Uhlenbeck type processes with different parameters, which in the context of continuous-time processes corresponds to the randomization of the Ornstein-Uhlenbeck type process parameter by using the integration with respect to a special type of random measure. These processes are first introduced in [2] and later generalized to the multivariate case in [6]. The motivation for their definition was the fact that the Ornstein-Uhlenbeck type process, despite being a powerful modeling tool, is not suitable for the case when empirical data show significant dependence on long time periods. Unlike the Ornstein-Uhlenbeck type process, whose correlation function decays exponentially fast, the correlation function of the supOU process can decay more slowly and, in some cases, exhibit the property of long-range dependence. This provides the possibility of modeling a larger class of phenomena. Additionally, the advantage of this process is that its marginal distribution and dependence structure can be modeled independently, providing greater flexibility in modeling. While the dependence structure depends on the choice of the probability measure used for randomizing the parameter, the marginal distribution of the process is determined by the Lévy process driving the Ornstein-Uhlenbeck type process. These processes have numerous applications, particularly in finance, where they serve as models for stochastic volatility. More about volatility models can be found, for example, in [5], [7] and [24].

In Chapter 2, we define infinite divisibility and the Lévy process as concepts that form the foundation for constructing the theory of supOU processes. Also, we list some of their important properties. The connection between them is explained, and the Lévy-Khintchine formula is presented as the main tool used throughout the thesis. Selfdecomposable random variables are also defined as an example of infinitely divisible random variables with useful properties, closely related to Ornstein-Uhlenbeck type processes.

In Chapter 3, the Ornstein-Uhlenbeck type process is defined, and some of its basic properties are listed. It is explained that selfdecomposable random variables appear as the marginal distributions of these processes.

In Chapter 4, the concept of a homogeneous infinitely divisible independently scattered random measure, i.e. a Lévy basis, is introduced, along with integration with respect to such a measure. The integrability conditions are listed, after which the supOU process is defined. The cumulant functions of the marginal distributions of the supOU process are calculated, followed by the correlation function. It is explained when this process achieves the property of long-range dependence. As an example of a process important in applications, especially as a model for integrated volatility, the integrated supOU process is briefly presented.

In Chapter 5, all the previously mentioned terms are generalized to the multivariate case and the multivariate supOU process is defined. It is shown under which conditions it is well-defined and when it has finite moments of each order. Also, expressions for the expectation, variance and covariance of this process are derived.

Throughout the thesis we consider that all random variables and random processes are defined on a fixed probability space $(\Omega, \mathcal{F}, P)$.

2 | Lévy processes and infinite divisibility

In this chapter, we will define Lévy processes and terms closely related to them. We will state some of their important properties and express important results that demonstrate how such processes are connected to infinite divisibility. These concepts will be of great importance for defining Ornstein-Uhlenbeck type processes, as well as supOU processes.

2.1 Infinite divisibility

Definition 1. *We say that an $\mathbb{R}$-valued random variable Y is infinitely divisible (or that Y has an infinitely divisible distribution) if, for all $n \in \mathbb{N}$, there exists a sequence of i.i.d. random variables $Y_1^{(n)}, \dots, Y_n^{(n)}$ such that*

$$Y \stackrel{d}{=} \sum_{k=1}^{n} Y_k^{(n)}.$$

Infinitely divisible random variables can be characterised by their characteristic functions or cumulant functions. This important result is known as the Lévy-Khintchine formula and it will be useful later on. To express it, we first introduce concepts of characteristic functions, cumulant functions and Lévy measures.

Definition 2. *The characteristic function of a random variable Y (or its distribution) is the function $\varphi_Y : \mathbb{R} \to \mathbb{C}$ defined by*

$$\varphi_Y(\zeta) = \mathbb{E}e^{i\zeta Y}.$$

The cumulant (generating) function of a random variable Y is the function $\kappa_Y : \mathbb{R} \to \mathbb{C}$ defined by

$$\kappa_Y(\zeta) = C\{\zeta \ddagger Y\} = \log \mathbb{E}e^{i\zeta Y}.$$

Assuming it exists, the m-th cumulant of Y, for $m \in \mathbb{N}$, is denoted by $\kappa_Y^{(m)}$, and it satisfies

$$\kappa_Y^{(m)} = (-i)^m \frac{d^m}{d\zeta^m} \kappa_Y(\zeta)\big|_{\zeta=0}. \tag{2.1}$$

If κ_Y is analytic around the origin, then

$$\kappa_Y(\zeta) = \sum_{m=1}^{\infty} \frac{(i\zeta)^m}{m!} \kappa_Y^{(m)}. \tag{2.2}$$

For a stochastic process $Y = \{Y(t)\}$ we write $\kappa_Y(\zeta, t) = \kappa_{Y(t)}(\zeta)$, and for the cumulant function of the random variable $Y(1)$ we write $\kappa_Y(\zeta) = \kappa_Y(\zeta, 1)$. Similarly, for the random variable $Y(t)$, the m-th cumulant is denoted by $\kappa_Y^{(m)}(t)$ and $\kappa_Y^{(m)} = \kappa_Y^{(m)}(1)$.

Note that

$$\kappa_Y''(\zeta) = -\frac{1}{\varphi_Y^2(\zeta)} \left(\varphi_Y'(\zeta)\right)^2 + \frac{1}{\varphi_Y(\zeta)} \varphi_Y''(\zeta),$$

from which it follows that

$$\kappa_Y^{(2)} = -\kappa_Y''(0) = \frac{1}{\varphi_Y^2(0)} \left(\varphi_Y'(0)\right)^2 - \frac{1}{\varphi_Y(0)} \varphi_Y''(0) = \mathbb{E}Y^2 - (\mathbb{E}Y)^2 = \operatorname{Var} Y. \quad (2.3)$$

In the previous equality, we use the fact that $\mathbb{E}X^k = \frac{1}{i^k}\varphi^{(k)}(0)$.

Remark 1. *For a random vector* $\mathbf{Y}$, *the characteristic function* $\varphi_{\mathbf{Y}} : \mathbb{R}^d \to \mathbb{C}$ *is defined as* $\varphi_{\mathbf{Y}}(\mathbf{u}) = \mathbb{E}e^{i\mathbf{u}^T\mathbf{Y}}$ *and analogously, as in the one-dimensional case, the cumulant function* $\kappa_{\mathbf{Y}} : \mathbb{R}^d \to \mathbb{C}$ *is defined as* $\kappa_{\mathbf{Y}}(\mathbf{u}) = C\{\mathbf{u} \ddagger \mathbf{Y}\} = \log \mathbb{E}e^{i\mathbf{u}^T\mathbf{Y}}$. *Expressions* (2.1) *and* (2.2) *can also be generalized to the multidimensional case, and similarly to* (2.3), *it can be shown that*

$$\kappa_{(Y_1,Y_2)}^{(2)} = \operatorname{Cov}(Y_1, Y_2) \quad (2.4)$$

holds for a two-dimensional random vector $\mathbf{Y} = (Y_1, Y_2)$.

Definition 3. *We say that* μ *is a Lévy measure defined on* $\mathbb{R}$ *if it is a deterministic Radon measure*[1] *on* $\mathbb{R} \setminus \{0\}$ *such that* $\mu(\{0\}) = 0$ *and* $\int_{\mathbb{R}} \min\{1, x^2\}\, \mu(dx) < \infty$.

The following theorem gives the representation of cumulant functions for infinitely divisible random variables.

Theorem 1 (Lévy-Khintchine formula, see [23, Theorem 8.1])**.** *A real-valued random variable* Y *is infinitely divisible if there exists a triplet* (a, b, μ), *where* $a \in \mathbb{R}$, $b > 0$ *and* μ *is a Lévy measure defined on* $\mathbb{R}$, *such that*

$$C\{\zeta \ddagger Y\} = ia\zeta - \frac{b}{2}\zeta^2 + \int_{\mathbb{R}} \left(e^{i\zeta x} - 1 - i\zeta\tau(x)\right) \mu(dx). \quad (2.5)$$

Moreover, the triplet (a, b, μ) *is unique.*

Function τ in Theorem 1 is the *centering function*. Typical choice for the centering function τ is $\tau(x) = x\mathbf{1}_{[-1,1]}(x)$ or

$$\tau(x) = \begin{cases} x, & \text{if } |x| \leq 1, \\ -1, & \text{if } x < -1, \\ 1, & \text{if } x > 1. \end{cases}$$

Also, we call (a, b, μ) in Theorem 1 the *characteristic triplet*.

[1] A Radon measure is a Borel measure on the Hausdorff topological space, that is finite on all compact sets, outer regular on all Borel sets, and inner regular on open sets. More about Radon measures can be found in [11], Chapter 7.

Remark 2. *The converse of this statement holds true. If $a \in \mathbb{R}$, $b > 0$ and μ is a Lévy measure defined on $\mathbb{R}$, then there exists an infinitely divisible random variable Y whose cumulant function is given by (2.5).*

The concept of infinite divisibility and the Lévy-Khintchine formula can be generalized to random vectors and stochastic processes. The previous theorem for the case of random vectors states that a random vector $\mathbf{Y}$ is infinitely divisible if there exists a triplet $(\mathbf{a}, \Sigma, \mu)$ such that

$$C\{\mathbf{u} \ddagger \mathbf{Y}\} = i\mathbf{u}^T\mathbf{a} - \frac{1}{2}\mathbf{u}^T\Sigma\mathbf{u} + \int_{\mathbb{R}^d} \left(e^{i\mathbf{u}^T\mathbf{x}} - 1 - i\mathbf{u}^T\mathbf{x}\mathbf{1}_{[0,1]}(\|\mathbf{x}\|)\right)\mu(d\mathbf{x}),$$

where $\mathbf{a} \in \mathbb{R}^d$, Σ is a positive semi-definite $d \times d$ matrix, and μ is a Lévy measure on $\mathbb{R}^d$, i.e. a deterministic Radon measure on $\mathbb{R}^d \setminus \{\mathbf{0}\}$ such that $\mu(\{\mathbf{0}\}) = 0$ and $\int_{\mathbb{R}^d} \min\{1, \|\mathbf{x}\|^2\}\,\mu(d\mathbf{x}) < \infty$. A stochastic process is said to be infinitely divisible if all of its finite-dimensional distributions are infinitely divisible.

2.2 Lévy processes

Lévy processes form a fundamental class of stochastic processes with wide-ranging applications. They belong to the class of infinitely divisible stochastic processes and generalize the concepts of random walks and Brownian motion.

Definition 4. *A stochastic process $L = \{L(t),\ t \geq 0\}$ is said to be a Lévy process if*

(i) $L(0) = 0$ *a.s.,*

(ii) *L has independent increments, i.e. for all $n \in \mathbb{N}$ and $0 < t_0 < t_1 < \cdots < t_n$, random variables $L(t_0)$, $L(t_1) - L(t_0)$, $L(t_2) - L(t_1)$, $\ldots$, $L(t_n) - L(t_{n-1})$ are independent,*

(iii) *L has stationary increments, i.e. for $0 \leq t < s$*

$$L(t) - L(s) \stackrel{d}{=} L(t-s),$$

(iv) *L is stochastically continuous, i.e. for all $\varepsilon > 0$ and for all $s \geq 0$*

$$\lim_{t \to s} P(|L(t) - L(s)| > \varepsilon) = 0,$$

(v) *The paths of L are P-almost surely right-continuous with left limits (càdlàg paths).*

As we mentioned earlier, Lévy processes are closely related to infinitely divisible random variables. Let $L = \{L(t),\ t \geq 0\}$ be a Lévy process. For each $n \in \mathbb{N}$ and $t \geq 0$ we can write

$$L(t) \stackrel{d}{=} \sum_{k=1}^{n} Y_k^{(n)}(t),$$

where

$$Y_k^{(n)}(t) = L\left(\frac{kt}{n}\right) - L\left(\frac{(k-1)t}{n}\right).$$

Due to the independence and stationarity of the increments, $Y_k^{(n)}(t)$ are i.i.d. So, we can conclude that $L(t)$ is an infinitely divisible random variable for each $t \geq 0$. It can be shown that the following statement holds true for Lévy processes.

Proposition 1 (see [1, Theorem 1.3.3])**.** *If L is a Lévy process, then*

$$\kappa_{L(t)}(\zeta) = t\kappa_L(\zeta)$$

for each $\zeta \in \mathbb{R}$, $t \geq 0$, where κ_L is the cumulant function of $L(1)$.

Since $L(t)$ is an infinitely divisible random variable for each $t \geq 0$, we know that for each $t \geq 0$ there exists a corresponding Lévy-Khintchine triplet. The previous proposition states that we can determine the characteristic triplet of the Lévy process at any time $t > 0$ if we know the characteristic triplet of $L(1)$. Furthermore, Theorem 2 tells us that for any infinitely divisible random variable Y, there exists a corresponding Lévy process $\{L(t),\ t \geq 0\}$ such that $L(1) \stackrel{d}{=} Y$. These results are important because they indicate that the law of the process $\{L(t),\ t \geq 0\}$ is determined by the law of $L(1)$.

Theorem 2 (Lévy-Khintchine formula for Lévy processes, see [18, Theorem 1.6])**.** *Suppose that $a \in \mathbb{R}$, $b > 0$ and μ is a Lévy measure defined on $\mathbb{R}$. For each $\zeta \in \mathbb{R}$ define*

$$\kappa_L(\zeta) = ia\zeta - \frac{b}{2}\zeta^2 + \int_{\mathbb{R}} \left(e^{i\zeta x} - 1 - i\zeta\tau(x)\right)\mu(dx).$$

Then there exists a probability space $(\Omega, \mathcal{F}, P)$, on which a Lévy process L is defined, such that $L(1)$ has the cumulant function $\kappa_L(\zeta)$.

Remark 3. *We say that L from the previous theorem is a Lévy process generated by κ_L (or by the infinitely divisible random variable that has the characteristic triplet (a, b, μ)). The cumulant function of the random variable $L(1)$ is often referred to as the cumulant function of the Lévy process L. Also, it is often said that (a, b, μ) is the characteristic triplet of the Lévy process L.*

2.3 Selfdecomposability

Another important concept is the selfdecomposability of the random variable. Such random variables are closely related to Ornstein-Uhlenbeck type processes and supOU processes, because they appear as their marginal distributions, as we will see later.

Definition 5. *An infinitely divisible random variable Y (or its distribution) is selfdecomposable if its characteristic function $\varphi_Y(\zeta) = \mathbb{E}e^{i\zeta Y}$, $\zeta \in \mathbb{R}$, has the property that for every $c \in (0,1)$ there exists a characteristic function φ_c such that*

$$\varphi_Y(\zeta) = \varphi_Y(c\zeta)\varphi_c(\zeta), \quad \forall \zeta \in \mathbb{R}.$$

Let Y be a selfdecomposable random variable. Then, Y has the same distribution as $cY + Z_c$, where Y and Z_c are independent, and Z_c has the characteristic function φ_c, $c \in (0,1)$. In this case, Y can be represented as

$$Y = \int_0^\infty e^{-s} dL(s), \tag{2.6}$$

where $L = \{L(t),\, t \geq 0\}$ is a Lévy process whose law is determined uniquely by that of Y (see [17], Theorem 3.6.8 and Theorem 3.9.3). The process L is called the *background driving Lévy process* (BDLP) corresponding to the infinitely divisible random variable Y. The cumulant functions of Y and $L(1)$ are related by (see [17], Remark 3.6.9)

$$\kappa_Y(\zeta) = \int_0^\infty \kappa_L(e^{-s}\zeta) ds. \tag{2.7}$$

Corollary 1 (see [16, Corollary 1]). *For a selfedecomposable random variable Y, κ_Y is differentiable for $\zeta \neq 0$, $\zeta \kappa_Y'(\zeta) \to 0$ as $0 \neq \zeta \to 0$ and*

$$\kappa_L(\zeta) = \zeta \kappa_Y'(\zeta),$$

where κ_L is the cumulant function of the corresponding BDLP.

The BDLP L can be extended to a two-sided Lévy process by putting for $t < 0$, $L(t) = -\widetilde{L}(-t-)$, where $\{\widetilde{L}(t),\, t \geq 0\}$ is an independent copy of the process $\{L(t),\, t \geq 0\}$ modified to be càdlàg.

3 | Ornstein–Uhlenbeck type processes

In this chapter, we will define Ornstein-Uhlenbeck type processes and state some of their important properties. These processes are important for defining supOU processes, which are essentially superpositions of Ornstein-Uhlenbeck type processes, as we will see later on.

Definition 6. *Let L be a two-sided Lévy process satisfying $\mathbb{E}\log(1+|L(1)|)<\infty$ and $\lambda>0$. The Ornstein-Uhlenbeck (OU) type process is a process $X=\{X(t),\, t\in\mathbb{R}\}$ defined by*

$$X(t)=X^{(\lambda,L)}(t)=e^{-\lambda t}\int_{-\infty}^{t}e^{\lambda s}dL(\lambda s)=\int_{\mathbb{R}}e^{-\lambda(t-s)}\mathbf{1}_{[0,\infty)}(t-s)\,dL(\lambda s). \quad (3.1)$$

The scaling of time in (3.1) ensures that the marginal distributions of the process do not depend on λ. The change of variables $s=\lambda s$ in (3.1) gives a second representation of X

$$X(t)=\int_{\mathbb{R}}e^{-\lambda t+s}\mathbf{1}_{[0,\infty)}(\lambda t-s)\,dL(s). \quad (3.2)$$

Note that the integration is with respect to a two-sided Lévy process. More about this type of integration can be found in [1], Chapter 4.

It can be shown that $X=\{X(t),\, t\in\mathbb{R}\}$ is strictly stationary (see [23]). Also, we can write

$$X(t)=e^{-\lambda t}\int_{-\infty}^{0}e^{\lambda s}\,dL(\lambda s)+e^{-\lambda t}\int_{0}^{t}e^{\lambda s}\,dL(\lambda s)\overset{d}{=}e^{-\lambda t}X(0)+e^{-\lambda t}\int_{0}^{t}e^{\lambda s}\,dL(\lambda s),$$

with the terms on the right-hand side independent. This shows that stationary distribution of $X=\{X(t), t\in\mathbb{R}\}$ is selfdecomposable distribution which corresponds to the BDLP L. That $X(0)$ corresponds to L follows from (2.6). The converse also holds. The following result indicates that for every selfdecomposable distribution, there exists a corresponding Ornstein-Uhlenbeck type process for which it is the stationary distribution.

Theorem 3 (see [5, Theorem 2.1]). *If Y is selfdecomposable, then there exists a stationary process $\{X(t), t\in\mathbb{R}\}$ and a Lévy process $\{L(t),\, t\geq 0\}$ such that $X(t)\overset{d}{=}Y$ and*

$$X(t)=\int_{-\infty}^{t}e^{-\lambda(t-s)}\,dL(\lambda s).$$

An Ornstein-Uhlenbeck type process can also be defined as a stationary solution of the stochastic differential equation

$$dX(t) = -\lambda X(t)dt + dL(\lambda t), \tag{3.3}$$

where L is a Lévy process satisfying $\mathbb{E}\log(1 + |L(1)|) < \infty$ and $\lambda > 0$. Condition $\mathbb{E}\log(1 + |L(1)|) < \infty$ is necessary for the existence of the stationary solution of the stochastic differential equation (3.3) (see [17], Theorem 3.6.6). For more details, see [23].

Remark 4. *It should be noted that the Ornstein-Uhlenbeck type process differs from the Ornstein-Uhlenbeck process. The Ornstein-Uhlenbeck process is a special case of an Ornstein-Uhlenbeck type process when the BDLP is a Brownian motion. More precisely, it is the solution to the Langevin stochastic differential equation*

$$dX(t) = \lambda(\mu - X(t)) + \sigma dB(t),$$

where B is a Brownian motion, $\mu \in \mathbb{R}$ and $\lambda, \sigma > 0$.

If we assume that X is square integrable, then the autocorrelation function of X is

$$r(\tau) = e^{-\lambda\tau}, \quad \tau \geq 0.$$

This follows from the fact that

$$X(t+\tau) = e^{-\lambda\tau}\left(X(t) + e^{-\lambda t}\int_t^{t+\tau} e^{\lambda s}\, dL(\lambda s)\right).$$

Thus, the autocorrelation function of an OU type process is exponential, which often does not match the dependence structure exhibited by empirical data, making it unsuitable for many applications.

OU type process may be seen as a continuos time analog of AR(1) process (CAR(1) process) since

$$\begin{aligned} X(n) &= e^{-\lambda}e^{-\lambda(n-1)}\int_{-\infty}^{n-1} e^{\lambda s}\, dL(\lambda s) + e^{-\lambda n}\int_{n-1}^{n} e^{\lambda s}\, dL(\lambda s) \\ &= e^{-\lambda}X(n-1) + e^{-\lambda n}\int_{n-1}^{n} e^{\lambda s}\, dL(\lambda s), \end{aligned}$$

and $\{e^{-\lambda n}\int_{n-1}^{n} e^{\lambda s}\, dL(\lambda s),\ n \in \mathbb{N}\}$ is stationary.

4 | SupOU processes

As previously mentioned, supOU processes are essentially superpositions of Ornstein-Uhlenbeck type processes, which will be shown to exhibit a more flexible dependence structure. Suppose that $\{X^{(\lambda_k,L)}(t),\, t \in \mathbb{R}\}$, $k = 1,\dots,m$ are independent OU type processes with parameters λ_k, $k = 1\dots,m$ and BDLP L. Then the finite superposition

$$X_m(t) = \sum_{k=1}^{m} w_k' X^{(\lambda_k,L)}(t) \tag{4.1}$$

has autocorrelation function

$$r(\tau) = \sum_{k=1}^{m} w_k e^{-\lambda_k|\tau|}, \quad \tau \in \mathbb{R}. \tag{4.2}$$

By extending the idea in (4.1) to an infinite superposition ($m = \infty$), one could obtain, for example, by putting $w_k = k^{-(1+\alpha)}$, $\lambda_k = \lambda/k$ for $\alpha, \lambda > 0$, that the autocorrelation function can decay to 0 more slowly than an exponential function. Alternatively, one can view superposition as averaging over randomized λ according to some probability measure π. In that sense, we will formally define the supOU process as

$$X(t) = \int_{\mathbb{R}_-}\int_{\mathbb{R}} e^{\xi(t-s)}\mathbf{1}_{[0,\infty)}(t-s)dL(s)\pi(d\xi),$$

which should be compared with (3.1). In order to give meaning to the previous integral, we need the concept of a random measure, Lévy basis and the integral with respect to a random measure.

4.1 Infinitely divisible random measures

Definition 7. *Let $(S,\mathcal{S})$ be a measurable space. A collection $\Lambda = \{\Lambda(A), A \in \mathcal{S}\}$ of random variables on some probability space $(\Omega,\mathcal{F},P)$ is said to be a random measure on $(S,\mathcal{S})$ if*

(i) $\Lambda(\emptyset) = 0$ *a.s.,*

(ii) for every sequence $\{A_n, n \in \mathbb{N}\}$ of disjoint sets in $\mathcal{S}$, it holds that

$$\Lambda\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \Lambda(A_n) \quad a.s.,$$

whenever $\bigcup_{n=1}^{\infty} A_n \in \mathcal{S}$.

We say that the random measure is independently scattered if for every sequence $\{A_n, n \in \mathbb{N}\}$ *of disjoint sets in* $\mathcal{S}$*, the random variables* $\Lambda(A_n)$, $n \in \mathbb{N}$*, are independent. We say that* Λ *is an infinitely divisible random measure, if all finite-dimensional distributions of* Λ *are infinitely divisible.*

Remark 5. *Alternatively, one may take* $\mathcal{S}$ *to be a* σ*-ring of* S*, i.e. countable unions of sets in* $\mathcal{S}$ *belong to* $\mathcal{S}$ *and if* A *and* B *are sets in* $\mathcal{S}$ *with* $A \subset B$*, then* $B \backslash A \in \mathcal{S}$ *(see [22]).*

Suppose that Λ is infinitely divisible independently scattered random measure. Then, for each $A \in \mathcal{S}$, $\Lambda(A)$ is an infinitely divisible random variable. From the Lévy-Khintchine representation, it follows that there exists $m_0(A) \in \mathbb{R}$, $m_1(A) \geq 0$ and a Lévy measure $Q_A(\cdot)$ on $\mathbb{R}$ such that

$$C\{\zeta \ddagger \Lambda(A)\} = i\zeta m_0(A) - \frac{\zeta^2}{2} m_1(A) + \int_{\mathbb{R}} \left(e^{i\zeta x} - 1 - i\zeta\tau(x)\right) Q_A(dx). \quad (4.3)$$

Proposition 2 (see [22, Proposition 2.1])**.** *(i) If* Λ *is infinitely divisible independently scattered random measure, then* $m_0 : \mathcal{S} \to \mathbb{R}$ *is a signed measure*[1]*,* $m_1 : \mathcal{S} \to [0, \infty)$ *is a measure and* $A \mapsto Q_A(B)$ *is a measure for every* $B \in \mathcal{B}(\mathbb{R} \setminus \{0\})$*. Moreover, there exists a unique* σ*-finite measure* Q *on* $\mathcal{S} \times \mathcal{B}(\mathbb{R})$ *such that*

$$Q(A \times B) = Q_A(B), \quad A \in \mathcal{S}, \quad B \in \mathcal{B}(\mathbb{R} \setminus \{0\}).$$

(ii) If m_0*,* m_1 *and* $Q_A(\cdot)$ *satisfy the conditions in (i), then there is a unique (in the sense of finite-dimensional distributions) infinitely divisible independently scattered random measure such that (4.3) holds.*

Proposition 2 tells us that there is a one-to-one correspondence between infinitely divisible independently scattered random measures and the class of parameters m_0, m_1 and Q. In this case we say that Λ has the Lévy characteristics (m_0, m_1, Q) and Q is called the *generalized Lévy measure*.

For $A \in \mathcal{S}$ we define

$$m(A) = |m_0|(A) + m_1(A) + \int_{\mathbb{R}} \min\left\{1, x^2\right\} Q_A(dx).$$

The measure m is called the *control measure* since $m(A) = 0$ if and only if $\Lambda(A') = 0$ a.s. for all $A' \subseteq A$ (see [22]). This measure is important for characterizing the class of deterministic functions that are integrable with respect to the random measure Λ.

Another parametrization of the infinitely divisible independently scattered random measures may be obtained by defining the *local characteristics* $a(s) = \frac{dm_0}{dm}(s)$,

[1] A signed measure generalizes the concept of a measure by allowing it to take on negative values as well as positive values, while still satisfying the property of σ-additivity. More about signed measures can be found in [11], Chapter 3.

$b(s) = \frac{dm_1}{dm}(s)$ and by disintegrating measure Q so that $Q(ds, dx) = \rho(s, dx)m(dx)$, where $\rho : S \times \mathcal{B}(\mathbb{R} \setminus \{0\}) \to [0, \infty]$ is a Lévy measure on $\mathcal{B}(\mathbb{R} \setminus \{0\})$. In this case, the cumulant function (4.3) may be written as

$$C\{\zeta \ddagger \Lambda(A)\} = \int_A \left(i\zeta a(s) - \frac{\zeta^2}{2} b(s) + \int_{\mathbb{R}} \left(e^{i\zeta x} - 1 - i\zeta\tau(x) \right) \rho(s, dx) \right) m(ds).$$

An infinitely divisible independently scattered random measure Λ is *homogeneous* if for $A \in \mathcal{S}$

$$m_0(A) = am(A), m_1(A) = bm(A) \text{ and } Q_A(\cdot) = m(A)\mu_L(\cdot),$$

for some Lévy-Khintchine triplet (a, b, μ_L). In terms of the local characteristics Λ is homogeneous if $a(s) = a$, $b(s) = b$ and $\rho(s, \cdot) = \mu_L(\cdot)$. Hence, to define a homogeneous infinitely divisible independently scattered random measure one only needs to specify the Lévy-Khintchine triplet and the control measure m. Note that m and μ_L are deterministic.

4.2 Integration with respect to a random measure

Let $(S, \mathcal{S})$ be a measurable space and Λ an infinitely divisible independently scattered random measure with control measure m. The integration of a function on S with respect to the random measure Λ can be defined first for real simple functions, then as a limit in probability of such integrals. This type of integral is important because, by integrating with respect to a random measure, various infinitely divisible processes can be obtained, that is, processes whose finite-dimensional distributions are all infinitely divisible. The conditions for integrability of functions with respect to Λ can be found in [22]. Only the most important results, which will be useful later, are stated here.

Definition 8. *Let Λ be an infinitely divisible random measure and f a real simple function on S, i.e. $f = \sum_{j=1}^n c_j \mathbf{1}_{A_j}$, where $c_1, \ldots, c_n \in \mathbb{R}$ and $A_1, \ldots, A_n \in \mathcal{S}$ are disjoint. For each $A \in \mathcal{S}$, the integral of the function f with respect to the random measure Λ is defined as*

$$\int_A f \, d\Lambda = \sum_{j=1}^n c_j \Lambda(A \cap A_j).$$

The previous definition discusses the integration of simple functions with respect to a random measure. The following definition defines the integral with respect to a random measure for general measurable functions.

Definition 9. *A measurable function $f : (S, \mathcal{S}) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ is said to be Λ-integrable if there exists a sequence $\{f_n, n \in \mathbb{N}\}$ of simple functions such that*

(i) $f_n \to f$ a.e. with respect to m,

(ii) for every $A \in \mathcal{S}$, the sequence $\{\int_A f_n \, d\Lambda, n \in \mathbb{N}\}$ converges in probability as $n \to \infty$.

If f is Λ-integrable, then

$$\int_A f\, d\Lambda = P - \lim_{n\to\infty} \int_A f_n\, d\Lambda,$$

where $P - \lim_{n\to\infty}$ denotes the limit in probability.

Note that $\int_A f_n\, d\Lambda$ in the previous theorem is a random variable for every $n \in \mathbb{N}$, so $\int_A f\, d\Lambda$ is also a random variable as the limit of random variables. The following proposition gives the expression for the cumulant function of $\int_S f\, d\Lambda$.

Proposition 3 (see [22, Proposition 2.6])**.** *If function f is integrable with respect to the random measure Λ, then the cumulant function of the random variable $\int_S f\, d\Lambda$ is*

$$\mathrm{C}\left\{\zeta \ddagger \int_S f\, d\Lambda\right\} = \int_S \kappa_L(\zeta f(\omega))\, m(d\omega), \tag{4.4}$$

where κ_L is the cumulant function associated with Λ.

From (4.4), it follows that if Λ is homogenous infinitely divisible independently scattered random measure with the characteristic triplet (a, b, μ_L), than the cumulant function of the random variable $\Lambda(A)$ is

$$\mathrm{C}\{\zeta \ddagger \Lambda(A)\} = m(A)\kappa_L(\zeta),$$

where κ_L is the cumulant function associated with the triplet (a, b, μ_L), i.e.

$$\kappa_L(\zeta) = i\zeta a - \frac{\zeta^2}{2} b + \int_{\mathbb{R}} \left(e^{i\zeta x} - 1 - i\zeta x \mathbf{1}_{[-1,1]}(x)\right) \mu_L(dx).$$

In this context, we will refer to the homogeneous infinitely divisible independently scattered random measure Λ as the *Lévy basis*. Note that to each Lévy basis Λ corresponds an associated Lévy process L. This process is referred to as the *underlying Lévy process*.

Remark 6. *It can be shown that any Lévy process with real values can be understood as a random measure on the corresponding measurable space (see [20, Theorem 2.1]). In the following, the case where $(S, \mathcal{S}) = (\mathbb{R}_- \times \mathbb{R}, \mathcal{B}(\mathbb{R}_- \times \mathbb{R}))$ will be particularly important. In this case, the underlying Lévy process is defined by*

$$L(t) = \Lambda(\mathbb{R}_- \times (0, t]), \quad L(-t) = \Lambda(\mathbb{R}_- \times (-t, 0)), \quad t \geq 0,$$

where Λ is a Lévy basis on $(\mathbb{R}_- \times \mathbb{R}, \mathcal{B}(\mathbb{R}_- \times \mathbb{R}))$. Another important example is when Λ is a Lévy basis on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. Then the underlying Lévy process is defined by (see [6, Section 2])

$$L(t) = \Lambda\left((0, t]\right), \quad L(-t) = \Lambda\left((-t, 0)\right), \quad t \geq 0,$$

This example shows that the integral defining the OU type process (3.1) can be understood as an integral with respect to a random measure, where the random measure is generated by the Lévy process.

It can be shown that the integral of a function with respect to an infinitely divisible random measure is an infinitely divisible random variable. The following theorem provides necessary and sufficient conditions for the existence of $\int_S f d\Lambda$ as well as the explicit expressions for the characteristic triplet of that random variable.

Proposition 4 (see [22, Theorem 2.7]). *Let $f : S \to \mathbb{R}$ be a measurable function, Λ infinitely divisible independently scattered random measure with control measure m and define*

$$U(u,s) = ua(s) + \int_{\mathbb{R}} (\tau(xu) - u\tau(x))\, \rho(s,dx),$$
$$V(u,s) = \int_{\mathbb{R}} \min\{1, |xu|^2\}\, \rho(s,dx).$$

Then f is Λ-integrable if and only if

(i) $\int_S |U(f(s),s)|\, m(ds) < \infty$,

(ii) $\int_S |f(s)|^2 b(s)\, m(ds) < \infty$,

(iii) $\int_S V(f(s),s)|\, m(ds) < \infty$.

In this case, the characteristic function of $\int_S f d\Lambda$ is given by

$$\mathbb{E}e^{i\zeta \int_S f d\Lambda} = \exp\left\{ia_f\zeta - \frac{b_f}{2}\zeta^2 + \int_{\mathbb{R}} \left(e^{i\zeta x} - 1 - i\zeta\tau(x)\right) \mu_f(dx)\right\}, \quad \zeta \in \mathbb{R},$$

where

$$a_f = \int_S U(f(s),s)\, m(ds),$$
$$b_f = \int_S |f(s)|^2 b(s)\, m(ds),$$
$$\mu_f(B) = Q\left(\{(s,x) \in S \times \mathbb{R} : f(s)x \in B \setminus \{0\}\}\right), \quad B \in \mathcal{B}(\mathbb{R}).$$

4.3 Definition

After introducing all the necessary concepts, we can finally provide a precise definition of the univariate supOU process, which will be generalized to the multivariate case in the following chapter.

Definition 10. *Let $(S, \mathcal{S}) = (\mathbb{R}_- \times \mathbb{R}, \mathcal{B}(\mathbb{R}_- \times \mathbb{R}))$, (a,b,μ) is a Lévy-Khintchine triplet such that*

$$\int_{|x|>1} \log|x|\, \mu(dx) < \infty, \tag{4.5}$$

and π is a probability measure on $\mathbb{R}_-$ such that

$$\int_{\mathbb{R}_-} |\xi|^{-1}\, \pi(d\xi) < \infty. \tag{4.6}$$

Let Λ be a Lévy basis, i.e. homogeneous infinitely divisible independently scattered random measure on $(\mathbb{R}_- \times \mathbb{R}, \mathcal{B}(\mathbb{R}_- \times \mathbb{R}))$, with control measure $m = \pi \times \mathrm{Leb}$ and $f_t : \mathbb{R}_- \times \mathbb{R} \to \mathbb{R}$ given by

$$f_t(\xi, s) = e^{\xi(t-s)} \mathbf{1}_{(-\infty,t]}(s) = e^{\xi(t-s)} \mathbf{1}_{[0,\infty)}(t-s).$$

The infinitely divisible process $X = \{X(t),\ t \in \mathbb{R}\}$ defined by

$$X(t) = \int_{\mathbb{R}_- \times \mathbb{R}} f_t(\xi, s)\, \Lambda(d\xi, ds). \tag{4.7}$$

is called a supOU process (a superposition of Ornstein-Uhlenbeck type processes).

The integral is well-defined and the process is strictly stationary (see Theorem 4 below). Using formula (4.4), it can be easily calculated that the cumulant function of the finite-dimensional distributions of the supOU process (4.7) is given by

$$\begin{aligned}
&C\left\{\zeta_1, \ldots, \zeta_m \ddagger (X(t_1), \ldots, X(t_m))\right\} \\
&= \int_{\mathbb{R}_-} \int_{\mathbb{R}} \kappa_L \left(\sum_{j=1}^{m} \mathbf{1}_{[0,\infty)}(t_j - s) \zeta_j e^{\xi(t_j - s)} \right) ds\, \pi(d\xi),
\end{aligned} \tag{4.8}$$

where $t_1 < \cdots < t_m$.

From (4.8) it follows that

$$\begin{aligned}
C\left\{\zeta \ddagger X(t)\right\} &= \int_{\mathbb{R}_-} \int_{\mathbb{R}} \kappa_L \left(\mathbf{1}_{[0,\infty)}(t-s) \zeta e^{\xi(t-s)} \right) ds\, \pi(d\xi) \\
&= \int_{\mathbb{R}_-} \int_{-\infty}^{t} \kappa_L \left(\zeta e^{\xi(t-s)} \right) ds\, \pi(d\xi).
\end{aligned} \tag{4.9}$$

From the previous expression we can see the stationarity as follows

$$\begin{aligned}
C\left\{\zeta \ddagger X(t+h)\right\} &= \int_{\mathbb{R}_-} \int_{-\infty}^{t+h} \kappa_L \left(\zeta e^{\xi(t+h-s)} \right) ds\, \pi(d\xi) \\
&= \int_{\mathbb{R}_-} \int_{-\infty}^{t+h} \kappa_L \left(\zeta e^{\xi(t-(s-h))} \right) ds\, \pi(d\xi) \\
&= \int_{\mathbb{R}_-} \int_{-\infty}^{t} \kappa_L \left(\zeta e^{\xi(t-u)} \right) du\, \pi(d\xi) \\
&= C\left\{\zeta \ddagger X(t)\right\}.
\end{aligned}$$

Furthermore, using the substitution $\xi(t-s) = -s$ in (4.9), we obtain

$$\begin{aligned}
C\left\{\zeta \ddagger X(t)\right\} &= -\int_{\mathbb{R}_-} \xi^{-1}\, \pi(d\xi) \int_0^\infty \kappa_L\left(\zeta e^{-s}\right) ds \\
&= \rho \int_0^\infty \kappa_L\left(\zeta e^{-s}\right) ds,
\end{aligned}$$

where $\rho := \int_{\mathbb{R}_-} |\xi|^{-1}\, \pi(d\xi)$. From (2.7), we now see that, up to a constant ρ, this is the cumulant function of the selfdecomposable random variable with BDLP

L, i.e. of the corresponding OU type process. Therefore, the one-dimensional marginal of a supOU process is a selfdecomposable distribution whose characteristic triplet may be expressed in terms of (a, b, μ) and depends on π only through the factor $\int_{\mathbb{R}_-} |\xi|^{-1}\pi(d\xi)$. We say that (a, b, μ, π) is the *generating quadruple* (see [10]). Note that the generating quadruple determines the law of the supOU process.

Remark 7. *In the first paper where these processes are introduced (see [2]), they are defined in a slightly different way. Let $S = \mathbb{R}_+ \times \mathbb{R}$, $\mathcal{S} = \mathcal{B}(S)$ and $m = \widetilde{\pi} \times Leb$ be the product of a probability measure $\widetilde{\pi}$ on $\mathbb{R}_+$ and the Lebesgue measure on $\mathbb{R}$. Let $\kappa_{\widetilde{X}}$ be the cumulant function of some selfdecomposable law and $(\widetilde{a}, \widetilde{b}, \widetilde{\mu}_L)$ be the characteristic triplet of the associated BDLP with cumulant function $\kappa_{\widetilde{L}}$. If we define the Lévy basis $\widetilde{\Lambda}$ on $\mathbb{R}_+ \times \mathbb{R}$ with generating quadruple $(\widetilde{a}, \widetilde{b}, \widetilde{\mu}_L, \widetilde{\pi})$, then the supOU process $\{\widetilde{X}(t),\ t \in \mathbb{R}\}$ is defined by (see [2, Theorem 3.1])*

$$\widetilde{X}(t) = \int_{\mathbb{R}_+} e^{-\xi t} \int_{-\infty}^{\xi t} e^s\, \widetilde{\Lambda}(d\xi, ds) = \int_{\mathbb{R}_+} \int_{\mathbb{R}} e^{-\xi t + s} \mathbf{1}_{[0,\infty)}(\xi t - s)\, \widetilde{\Lambda}(d\xi, ds). \quad (4.10)$$

This definition is different from the one in (4.7). Here $\widetilde{\Lambda}$ is a Lévy basis associated with control measure $\widetilde{\pi} \times$ Leb for any probability measure $\widetilde{\pi}$, not necessarily satisfying $\int_{\mathbb{R}_+} |\xi|^{-1}\widetilde{\pi}(d\xi) < \infty$. The two formulas are formally related by the change of variables, as shown in [10, Proposition 2.1]. In short, by taking $\widetilde{a} = \rho a$, $\widetilde{b} = \rho b$, $\widetilde{\mu}_L = \rho \mu_L$ and $\widetilde{\pi}(d\xi) = \rho^{-1}\xi^{-1}\pi(d\xi)$, where $\rho = \int_{\mathbb{R}_-} |\xi|^{-1}\pi(d\xi) < \infty$, we obtain a process which has the same law as the process X defined in (4.7). Also, (4.7) may be seen as a randomization of λ in (3.1), while (4.10) follows by randomizing λ in (3.2).

SupOU processes belong to the class of *mixed moving average processes*. For a continuous-time stochastic process $\{X(t),\ t \in \mathbb{R}\}$, we say that it is a mixed moving average process if it can be represented as

$$X(t) = \int_S f(\xi, t - s)\, \Lambda(d\xi, ds),$$

where Λ is an infinitely divisible independently scattered random measure and f is Λ-integrable function. For the supOU process (4.7) it holds that

$$X(t) = \int_{\mathbb{R}_- \times \mathbb{R}} \widetilde{f}(\xi, t - s)\, \Lambda(d\xi, ds),$$

with

$$\widetilde{f}(\xi, u) = e^{\xi u} \mathbf{1}_{[0,\infty)}(u).$$

Example 1. *Suppose that the probability measure π on $\mathbb{R}_-$ is degenerate such that $\pi(\{-\lambda\}) = 1$ for some $\lambda > 0$. Then we have that*

$$C\{\zeta_1, \dots, \zeta_m \ddagger (X(t_1), \dots, X(t_m))\} = \int_{\mathbb{R}} \kappa_L \left(\sum_{j=1}^{m} \mathbf{1}_{[0,\infty)}(t_j - s)\zeta_j e^{-\lambda(t_j - s)} \right) ds.$$

If we determine the joint cumulant function of the OU type process (3.1) using relation (4.8), we can see that the finite-dimensional distributions of the supOU process with this probability measure are the same as those for the OU type process, up to the factor $1/\lambda$.

Example 2. *Suppose π is a discrete probability measure on $\mathbb{R}_-$ such that $\pi(\{-\lambda_k\}) = p_k$, $k \in \mathbb{N}$ and $\lambda_k > 0$. Then condition $\int_{\mathbb{R}_-} |\xi|^{-1}\pi(d\xi) < \infty$ is equivalent to $\rho := \sum_{k=1}^{\infty} \lambda_k^{-1} p_k < \infty$ and we have that*

$$
\begin{aligned}
&C\{\zeta_1, \ldots, \zeta_m \ddagger (X(t_1), \ldots, X(t_m))\} \\
&= \sum_{k=1}^{\infty} \int_{\mathbb{R}} p_k \kappa_L \left(\sum_{j=1}^{m} \mathbf{1}_{[0,\infty)}(t_j - s)\zeta_j e^{-\lambda_k(t_j - s)} \right) ds.
\end{aligned}
$$

In this case X has the same distribution, up to the factor $1/\rho$, as the infinite discrete type superposition

$$
\left\{ \sum_{k=1}^{\infty} X^{(\lambda_k, L)}(t),\ t \in \mathbb{R} \right\},
$$

where $\{X^{(\lambda_k,L)}(t),\ t \in \mathbb{R}\}$, $k \in \mathbb{N}$ are independent OU type processes corresponding to parameter λ_k and BDLP L with cumulant function $p_k\kappa_L$, $k \in \mathbb{N}$. Note that this corresponds to (4.1) for $m = \infty$.

We have seen that the marginal distribution of the supOU process depends on the probability measure π only through the factor ρ, and below we will see that the correlation function of the supOU process depends only on the choice of π. Thus, for the supOU process, the marginal distribution and the dependence structure can be modeled independently. Different combinations of probability measures π and selfdecomposable distributions yield different supOU processes, which have that exact selfdecomposable distribution as their marginal distribution. For example, if we choose $-\Gamma(\alpha+1,1), \alpha > 0$ distribution for π and a selfdecomposable distribution such as the Inverse Gaussian, Normal Inverse Gaussian, or any stable distribution, we obtain different supOU processes with the same correlation function.

4.4 Dependence structure

As previously mentioned, supOU processes are important because they exhibit a much more flexible dependence structure compared to, for example, OU type processes. In some cases, they can exhibit a property of *long-range dependence* (also known as the *long memory property*), which is defined below. Briefly, for a square-integrable stationary random process, we say it has short-range dependence if its correlation function is integrable, and it has long-range dependence if it is not integrable. The correlation function of a process with long-range dependence should decay more slowly than exponential. The property of long-range dependence is important, for example, in time series of financial and economic data, where the sample autocorrelation function may have high values at large lags.

To calculate the correlation function of the supOU process (4.7), we use the fact that the second-order cumulant of a random vector is equal to the covariance between its components (see (2.4)). Equality (4.8) for $m = 2$ gives

$$\begin{aligned}
&C\left\{\zeta_1, \zeta_2 \ddagger (X(t_1), X(t_2))\right\} \\
&= \int_{\mathbb{R}_-} \int_{\mathbb{R}} \kappa_L \left(\mathbf{1}_{[0,\infty)}(t_1 - s)\zeta_1 e^{\xi(t_1-s)} + \mathbf{1}_{[0,\infty)}(t_2 - s)\zeta_2 e^{\xi(t_2-s)} \right) \Lambda(d\xi, ds).
\end{aligned}$$

By taking derivatives with respect to ζ_1 and ζ_2 and letting ζ_1, $\zeta_2 \to 0$, we obtain

$$\begin{aligned}
&\mathrm{Cov}(X(t_1), X(t_2)) \\
&= \kappa_L''(0) \int_{\mathbb{R}_-} \int_{\mathbb{R}} \mathbf{1}_{[0,\infty)}(t_1 - s) e^{\xi(t_1-s)} \mathbf{1}_{[0,\infty)}(t_2 - s) e^{\xi(t_2-s)} \Lambda(d\xi, ds).
\end{aligned}$$

Using relation (2.3) and the fact that due to $t_1 < t_2$ it holds that

$$\mathbf{1}_{[0,\infty)}(t_1 - s)\mathbf{1}_{[0,\infty)}(t_2 - s) = \mathbf{1}_{[0,\infty)}(t_1 - s),$$

we obtain

$$\begin{aligned}
\mathrm{Cov}(X(t_1), X(t_2)) &= \mathrm{Var}(L(1)) \int_{\mathbb{R}_-} \int_{\mathbb{R}} \mathbf{1}_{[0,\infty)}(t_1 - s) e^{\xi(t_1-s)+\xi(t_2-s)}\, ds\, \pi(d\xi) \\
&= \mathrm{Var}(L(1)) \int_{\mathbb{R}_-} \int_{-\infty}^{t_1} e^{\xi(t_1+t_2)-2\xi s}\, ds\, \pi(d\xi) \\
&= -\frac{1}{2} \mathrm{Var}(L(1)) \int_{\mathbb{R}_-} \xi^{-1} e^{\xi(t_1+t_2)} e^{-2\xi t_1}\, \pi(d\xi) \\
&= -\frac{1}{2} \mathrm{Var}(L(1)) \int_{\mathbb{R}_-} \xi^{-1} e^{\xi(t_2-t_1)}\, \pi(d\xi).
\end{aligned}$$

Furthermore, since the process is stationary, we have that $\mathrm{Cov}(X(t_1), X(t_2)) = \mathrm{Cov}(X(0), X(t_2 - t_1))$ and for $\tau > 0$ we can write

$$\mathrm{Cov}(X(0), X(\tau)) = -\frac{1}{2} \mathrm{Var}(L(1)) \int_{\mathbb{R}_-} \xi^{-1} e^{\xi\tau}\, \pi(d\xi) \tag{4.11}$$

Also, it holds that

$$\mathrm{Var}(X(\tau)) = \mathrm{Var}(X(0)) = -\frac{1}{2} \mathrm{Var}(L(1)) \int_{\mathbb{R}_-} \xi^{-1}\, \pi(d\xi) = \frac{\rho}{2} \mathrm{Var}(L(1)), \tag{4.12}$$

where $\rho = \int_{\mathbb{R}_-} |\xi|^{-1}\, \pi(d\xi)$. Finally, we obtain the autocorrelation function of the supOU process (4.7)

$$r(\tau) = \frac{\int_{\mathbb{R}_-} \xi^{-1} e^{\xi\tau}\, \pi(d\xi)}{\int_{\mathbb{R}_-} \xi^{-1}\, \pi(d\xi)} = -\rho^{-1} \int_{\mathbb{R}_-} \xi^{-1} e^{\xi\tau}\, \pi(d\xi), \tag{4.13}$$

Therefore, the dependence structure depends only on the choice of the probability measure π, and not on the characteristic triplet (a, b, μ). This implies that the existence of the long-range dependence property also depends on the choice of the measure π.

Example 3. *If we reconsider the degenerate case when π is a measure on $\mathbb{R}_-$ such that $\pi(\{-\lambda\}) = 1$ for some $\lambda > 0$, as in Example 1, it follows from relation (4.13) that*

$$r(\tau) = \rho^{-1}\lambda^{-1}e^{-\lambda\tau} = e^{-\lambda\tau},$$

since $\rho = \lambda^{-1}$ due to the degeneracy. We can see that this is exactly the correlation function of an OU type process.

Example 4. *From (4.13) the correlation function of the process from Example 2 is given by*

$$r(\tau) = \rho^{-1}\sum_{k=1}^{\infty}\lambda_k^{-1}e^{-\lambda_k\tau}p_k, \quad \tau \geq 0,$$

which corresponds to (4.2).

To define the long-range dependence property, we first need the concept of a slowly varying function at infinity. Also, we will use the notation $f \sim g$ if $f(x)/g(x) \to 1$ as $x \to 0$ or $x \to \infty$.

Definition 11. *We say that function ℓ is slowly varying at infinity if for every $x > 0$*

$$\lim_{t\to\infty}\frac{\ell(xt)}{\ell(t)} = 1.$$

Definition 12. *A stationary process with correlation function r exhibits long-range dependence, if there exists $\alpha \in (0,1)$ and a slowly varying function ℓ, such that*

$$r(\tau) \sim \ell(\tau)\tau^{-\alpha}, \quad \text{for } \tau \to \infty.$$

It can be observed that the property of long-range dependence implies that $\int_0^\infty r(\tau)\,d\tau = \infty$. The following proposition indicates that the probability measure π needs to have a sufficiently large mass around the origin. The larger this mass is near the origin, the slower is the decay of the correlation function at infinity.

Proposition 5 (see [10, Proposition 2.6])**.** *Suppose X is a square integrable supOU process with correlation function r, ℓ is a slowly varying function at infinity and $\alpha > 0$. If*

$$\pi((0,x]) \sim \rho(\alpha+1)^{-\alpha}\ell(x^{-1})x^{\alpha+1}, \quad \text{as } x \to 0, \tag{4.14}$$

then

$$r(\tau) \sim \Gamma(\alpha)\ell(\tau)\tau^{-\alpha}, \quad \text{as } \tau \to \infty. \tag{4.15}$$

The converse holds true if π is absolutely continuous with density π', and $x^{-1}\pi'(x)$ is monotone on $(0, x_0)$ for some $x_0 > 0$.

The previous result follows from Karamata's Tauberian theorem (see [9, Theorem 1.7.1′]). From (4.15), it can be observed that if $\alpha \in (0,1)$, the correlation function is not integrable. In that case, the supOU process exhibits long-range dependence. Otherwise, for $\alpha \geq 1$, it exhibits short-range dependence.

Example 5. *Suppose X is a supOU process such that π is $-\Gamma(\alpha+1,1)$ distribution with density*

$$f(x) = \frac{1}{\Gamma(\alpha+1)}(-x)^{\alpha} e^{x} \mathbf{1}_{(-\infty,0)}(x),$$

where $\alpha > 0$. It can be shown that π satisfies (4.14). We have

$$\begin{aligned}\rho &= \int_{\mathbb{R}_-} |\xi|^{-1}\,\pi(d\xi) = -\int_{\mathbb{R}_-} \xi^{-1} \frac{1}{\Gamma(\alpha+1)}(-\xi)^{\alpha} e^{\xi}\,d\xi \\ &= \frac{1}{\alpha}\int_{\mathbb{R}_-} \frac{1}{\Gamma(\alpha)}(-\xi)^{\alpha-1} e^{\xi}\,d\xi = \frac{1}{\alpha},\end{aligned}$$

where the last integral is equal to 1 due to the fact that $f(x) = \frac{1}{\Gamma(\alpha)}(-x)^{\alpha-1}e^{x}\mathbf{1}_{(-\infty,0)}(x)$ is the probability density function of the $-\Gamma(\alpha,1)$ distribution. From relation (4.13), we can explicitly compute that

$$\begin{aligned}r(\tau) &= -\rho^{-1}\int_{\mathbb{R}_-} \xi^{-1} e^{\xi\tau}\,\pi(d\xi) = -\alpha\int_{\mathbb{R}_-} \xi^{-1} e^{\xi\tau} \frac{1}{\Gamma(\alpha+1)}(-\xi)^{\alpha} e^{\xi}\,d\xi \\ &= \int_{\mathbb{R}_-} \frac{1}{\Gamma(\alpha)}(-\xi)^{\alpha-1} e^{\xi(\tau+1)}\,d\xi = (\tau+1)^{-\alpha}.\end{aligned}$$

The last equality follows from the fact that $f(x) = \frac{(\tau+1)^{\alpha}}{\Gamma(\alpha)}(-x)^{\alpha-1}e^{(\tau+1)x}\mathbf{1}_{(-\infty,0)}(x)$ is the probability density function of the $-\Gamma(\alpha,\tau+1)$ distribution. We can see that the correlation function has the property

$$r(\tau) \sim \tau^{-\alpha}, \quad \text{as } \tau \to \infty.$$

Therefore, we can conclude that for $\alpha \in (0,1]$, the correlation function exhibits the property of long-range dependence, while for $\alpha > 1$, it does not.

Other examples of different correlation functions of the supOU processes can be found in [4].

4.5 Integrated processes

In this subsection, we introduce integrated supOU processes. They are another important example of processes that find application in mathematical finance, where they serve as models for integrated volatility. For these processes, explicit expressions for the cumulant function and cumulants can be derived.

Let $\{X(t),\, t \in \mathbb{R}\}$ be a supOU process defined in (4.7). The *integrated supOU process* $\{X^*(t),\, t \in \mathbb{R}\}$ is defined as

$$X^*(t) = \int_0^t X(u)du.$$

The following lemma is important to show that the integrated process is well-defined. To prove the lemma, the stochastic Fubini theorem is needed (see [3, Theorem 3.1]).

Lemma 1 (see [14, Lemma 4.1]). *For the integrated supOU process X^* one has*

$$X^*(t) = \int_0^t \left(\int_{\mathbb{R}_- \times \mathbb{R}} f_u(\xi, s)\, \Lambda(d\xi, ds) \right) du$$
$$= \int_{\mathbb{R}_- \times \mathbb{R}} \left(\int_0^t f_u(\xi, s)\, du \right) \Lambda(d\xi, ds), \text{ a.s.}$$

where $f_u(\xi, s) = e^{\xi(t-s)} \mathbf{1}_{[0,\infty)}(t - s)$.

Using the previous lemma, we have

$$X^*(t) = \int_{\mathbb{R}_- \times \mathbb{R}} F_t(\xi, s)\, \Lambda(d\xi, ds)$$

where

$$F_t(\xi, s) = \int_0^t f_u(\xi, s)\, du = \int_0^t e^{\xi(u-s)} \mathbf{1}_{[0,\infty)}(u - s)\, du = \begin{cases} \int_0^t e^{\xi(u-s)}\, du, & s \leq 0, \\ \int_s^t e^{\xi(t-s)}\, du, & 0 < s < t, \\ 0, & s \geq t, \end{cases}$$
$$= \begin{cases} \xi^{-1}(e^{\xi(t-s)} - e^{-\xi s}), & s \leq 0, \\ \xi^{-1}(e^{\xi(t-s)} - 1), & 0 < s < t, \\ 0, & s \geq t. \end{cases}$$

Note that we can write

$$F_t(\xi, s) = g(\xi, t - s) - g(\xi, -s),$$

with

$$g(\xi, u) = \begin{cases} \xi^{-1}(e^{\xi u} - 1), & u > 0, \\ 0, & u \leq 0. \end{cases}$$

This shows that X is *stationary increment mixed moving average* (SIMMA) process. More information about SIMMA processes can be found in [8].

Now, let X be a supOU process and X^* the corresponding integrated process. To ensure the existence of all cumulants of the marginal distribution of the underlying supOU process, the cumulant function κ_X needs to be analytic. The following lemma provides a criterion for checking analyticity.

Lemma 2 (see [19, Theorem 7.2.1]). *The characteristic and cumulant functions are analytic in a neighborhood of the origin if and only if there is a constant C such that the corresponding distribution function F satisfies*

$$1 - F(x) + F(-x) = O(e^{-ux}), \quad \text{as } x \to \infty,$$

for all $0 < u < C$.

Lemma 2 implies that the cumulant function of $X(t)$ is analytic in the neighborhood of the origin if there exists $a > 0$ such that

$$\mathbb{E}e^{a|X(t)|} < \infty.$$

This implies that all the moments and cumulants of $X(t)$ exist (see [12]). It is also important to note that the analyticity of the cumulant function does not depend on the probability measure π, because the one-dimensional marginal distributions are independent of the choice of π.

The following two propositions are expressed in terms of $\widetilde{\pi}$, which corresponds to the alternative parametrization of the process X, as described in Remark 7. They provide the expressions for the cumulant function and the cumulants of the integrated process. Firstly, for $a, b \in \mathbb{R}$, let

$$\varepsilon(a,b) = \frac{1}{b}\left(1 - e^{-ab}\right).$$

Proposition 6 (see [2, Theorem 4.1]). *The cumulant function κ_{X^*} of $X^*(t)$ satisfies*

$$\kappa_{X^*}(\zeta,t) = \zeta \int_0^\infty \int_0^t \kappa'_{\widetilde{X}}(\varepsilon(s,\xi)\zeta)\, ds\, \widetilde{\pi}(d\xi),$$

where $\kappa_{\widetilde{X}}(\zeta)$ is the cumulant function of $\widetilde{X}(1)$.

Proposition 7 (see [2, Theorem 4.2]). *Assume that the cumulant function $\kappa_{\widetilde{X}}$ of $\widetilde{X}(1)$ is analytic in a neighborhood of the origin. The cumulants of $X^*(t)$ are then given by*

$$\kappa_{X^*}^{(m)}(t) = \kappa_{\widetilde{X}}^{(m)} m I_{m-1}(t),$$

where the $\kappa_{\widetilde{X}}^{(m)}$ are the cumulants of $\widetilde{X}(1)$,

$$I_{m-1}(t) = \int_0^\infty \left(a_{m-1} + t\xi + \sum_{k=1}^{m-1} (-1)^{k-1} \binom{m-1}{k} \frac{1}{k} e^{-kt\xi}\right) \xi^{-m}\, \widetilde{\pi}(d\xi)$$

with

$$a_{m-1} = \sum_{k=1}^{m-1} (-1)^k \binom{m-1}{k} \frac{1}{k}.$$

Example 6 (see [12, Example 6]). *Assume that $\widetilde{X}$ is a supOU process whose marginal distribution is $IG(\delta,\gamma)$, that is the distribution over $[0,\infty)$ with probability density function*

$$f(x) = \frac{\delta}{\sqrt{2\pi}} e^{\delta\gamma} x^{-3/2} \exp\left\{-\frac{1}{2}\left(\delta^2 x^{-1} + \gamma^2 x\right)\right\}, \quad 0 < x < \infty,$$

where $\delta > 0$ i $\gamma > 0$. Since exponential moments for this distribution are finite, the cumulant function is analytic around the origin and takes the form

$$\kappa_{\widetilde{X}}(\zeta) = \delta\left(\gamma - \sqrt{\gamma^2 - 2i\zeta}\right).$$

There are also examples of selfdecomposable distributions for which the cumulant function is not analytic around the origin. For instance, the Student's t-distribution $T(\nu, \delta, \mu)$, $\nu, \delta > 0, \mu \in \mathbb{R}$, satisfies $\mathbb{E}|X|^q = \infty$ for $q > \nu$, making it an example of such a distribution. For distributions like these, where moments exist only up to a certain order, expressions for the cumulants can still be derived.

It can be shown that this process may exhibit the *intermittency* property, which indicates that the moments of the process do not have typical limiting behavior and is often used to describe models that achieve a high degree of variability. More about this can be found in [12], [13] and [14].

5 | Multivariate supOU processes

In this chapter, we introduce multivariate supOU processes and provide conditions for their existence as well as the finiteness of moments. We will explicitly express second-order moments and provide examples of processes that exhibit long-range dependence. Multivariate supOU processes are important for applications because it is often necessary to model several related time series. They are obtained as a superposition of independent multivariate OU type processes, which are solutions to stochastic differential equations of the form

$$dX(t) = AX(t)dt + dL(t),$$

where L is a d-dimensional Lévy process and A is a $d \times d$ matrix.

First, it is necessary to introduce some notations. We denote the set of all real $m \times n$ matrices by $M_{mn}(\mathbb{R})$ and if $m = n$ we write $M_n(\mathbb{R})$. I_n denotes the $n \times n$ identity matrix. The group of invertible $n \times n$ matrices is denoted by $GL_n(\mathbb{R})$, $\mathbb{S}_n$ is the linear subspace of symmetric matrices, $\mathbb{S}_n^+$ is the (closed) positive semi-definite cone and $\mathbb{S}_n^{++}$ is the open positive definite cone. Similarly, $\mathbb{S}_n^{--}$ denotes the set of the negative definite matrices. The spectrum of a matrix A is denoted by $\sigma(A)$. A^T is the transpose of a matrix A and a_{ij} is the element of matrix A in the i-th row and j-th column.

To define multivariate supOU processes, it is necessary to generalize the concept of a Lévy basis as well as the integration with respect to a random measure. Let $M_d^- := \{X \in M_d(\mathbb{R}) : \sigma(X) \subset (-\infty, 0) + i\mathbb{R}\}$, that is the set of all d-dimensional matrices whose eigenvalues have a strictly negative real part, and let $\mathcal{B}_b(M_d^- \times \mathbb{R})$ to be the bounded Borel sets of $M_d^- \times \mathbb{R}$. In the multivariate case, we define the Lévy basis Λ as a homogeneous infinitely divisible independently scattered random measure on $(M_d^- \times \mathbb{R}, \mathcal{B}_b(M_d^- \times \mathbb{R}))$. Note that in this case, $\Lambda(A)$ for $A \in \mathcal{B}_b(M_d^- \times \mathbb{R}))$ is a $\mathbb{R}^d$-valued random vector. Similar to the one-dimensional case, for the Lévy basis we have

$$C\{\mathbf{u} \ddagger \Lambda(A)\} = \Pi(A)\kappa_L(\mathbf{u}), \tag{5.1}$$

for all $\mathbf{u} \in \mathbb{R}^d$ and $A \in \mathcal{B}_b(M_d^- \times \mathbb{R}))$. The measure Π is the product of a probability measure π on M_d^- and the Lebesgue measure on $\mathbb{R}$. Also, κ_L is the cumulant transform of an infinitely divisible distribution on $\mathbb{R}^d$ with characteristic triplet $(\mathbf{a}, \Sigma, \mu)$, that is

$$\kappa_L(\mathbf{u}) = i\mathbf{u}^T\mathbf{a} - \frac{1}{2}\mathbf{u}^T\Sigma\mathbf{u} + \int_{\mathbb{R}^d} \left(e^{i\mathbf{u}^T\mathbf{x}} - 1 - i\mathbf{u}^T\mathbf{x}\mathbf{1}_{[0,1]}(\|\mathbf{x}\|)\right)\mu(d\mathbf{x}),$$

where $\mathbf{a} \in \mathbb{R}^d, \Sigma \in \mathbb{S}_d^+$ and μ is a Lévy measure on $\mathbb{R}^d$. So, the generating quadruple of the Levy basis is $(\mathbf{a}, \Sigma, \mu, \pi)$. The corresponding underlying Lévy process with the characteristic triplet $(\mathbf{a}, \Sigma, \mu)$ and cumulant function κ_L is defined by

$$L(t) = \Lambda\left(M_d^- \times (0,t]\right), \quad L(-t) = \Lambda\left(M_d^- \times (-t,0)\right), \quad t \geq 0.$$

Integration with respect to a random measure can be easily generalized to the multivariate case. The integral with respect to a random measure is defined analogously to the one-dimensional case, first for simple functions, and then as the limit in probability of such integrals of simple functions. Similarly to the one-dimensional case, since $\Lambda(A)$ is an $\mathbb{R}^d$-valued random vector for $A \in \mathcal{B}_b(M_d^- \times \mathbb{R}))$, the integrals of simple functions are also $\mathbb{R}^d$-valued random vectors, and hence the integral of a general measurable function, as a limit of $\mathbb{R}^d$-valued random vectors, is also an $\mathbb{R}^d$-valued random vector. The following proposition is a generalization of Proposition 4.

Proposition 8 (see [6, Proposition 2.3])**.** *Let Λ be an $\mathbb{R}^d$-valued Lévy basis on $M_d^- \times \mathbb{R}$ with cumulant function of the form (5.1) and $f : M_d^- \times \mathbb{R} \to M_d(\mathbb{R})$ a $\mathcal{B}\left(M_d^- \times \mathbb{R}\right) - \mathcal{B}\left(M_d(\mathbb{R})\right)$ measurable function. Then f is Λ-integrable if and only if*

$$\begin{aligned}
&\int_{M_d^-}\int_{\mathbb{R}} \| f(A,s)\mathbf{a} \\
&\qquad + \int_{\mathbb{R}^d} f(A,s)\mathbf{x}\left(1_{[0,1]}(\|f(A,s)\mathbf{x}\|) - 1_{[0,1]}(\|\mathbf{x}\|)\right)\mu(d\mathbf{x})\| \, ds\, \pi(dA) < \infty, \\
&\int_{M_d^-}\int_{\mathbb{R}} \| f(A,s)\Sigma f(A,s)^T \| \, ds\, \pi(dA) < \infty, \\
&\int_{M_d^-}\int_{\mathbb{R}}\int_{\mathbb{R}^d} \left(1 \wedge \|f(A,s)\mathbf{x}\|^2\right)\mu(d\mathbf{x})\, ds\, \pi(dA) < \infty.
\end{aligned}$$

If f is Λ-integrable, the distribution of $\int_{M_d^-}\int_{\mathbb{R}_+} f(A,s)\Lambda(dA, ds)$ is infinitely divisible with characteristic function

$$\begin{aligned}
&E\left(\exp\left(i\mathbf{u}^T \int_{M_d^-}\int_{\mathbb{R}_+} f(A,s)\,\Lambda(dA,ds)\right)\right) \\
&\quad = \exp\left(\int_{M_d^-}\int_{\mathbb{R}_+} \kappa_L\left(f(A,s)^T\mathbf{u}\right) ds\, \pi(dA)\right)
\end{aligned}$$

and characteristic triplet $(\mathbf{a}_{\text{int}}, \Sigma_{\text{int}}, \mu_{\text{int}})$ given by

$$\begin{aligned}
\mathbf{a}_{\text{int}} &= \int_{M_d^-}\int_{\mathbb{R}} (f(A,s)\mathbf{a} \\
&\quad + \int_{\mathbb{R}^d} f(A,s)\mathbf{x}(\mathbf{1}_{[0,1]}(\|f(A,s)\mathbf{x}\|) - \mathbf{1}_{[0,1]}(\|\mathbf{x}\|))\,\mu(d\mathbf{x}))\, ds\, \pi(dA), \\
\Sigma_{\text{int}} &= \int_{M_d^-}\int_{\mathbb{R}} f(A,s)\Sigma f(A,s)^T\, ds\, \pi(dA), \\
\mu_{\text{int}}(B) &= \int_{M_d^-}\int_{\mathbb{R}}\int_{\mathbb{R}^d} \mathbf{1}_B(f(A,s)\mathbf{x})\,\mu(d\mathbf{x})\, ds\, \pi(dA), \quad \forall B \in \mathcal{B}(\mathbb{R}^d).
\end{aligned}$$

Remark 8. *Note that in the integrals above, both the integrands and the integrators are matrix (or vector) valued. In short, if we consider* $\{A(t), t \geq 0\}$ *in* $M_{mn}(\mathbb{R})$ *and* $\{B(t), t \geq 0\}$ *in* $M_{pq}(\mathbb{R})$ *as càdlàg and adapted processes, and* $\{L(t), t \geq 0\}$ *in* $M_{np}(\mathbb{R})$ *as a Lévy process, then with* $\int_0^t A(s-)dL(s)B(s-)$ *we denote a matrix* $C(t)$ *in* $M_{mq}(\mathbb{R})$ *such that its element in the i-th row and j-th column is* $c_{ij}(t) = \sum_{k=1}^{n} \sum_{l=1}^{p} \int_0^t a_{ik}(s-)b_{lj}(s-)dL_{kl}(s)$. *More about this type of integration can be found in* [21], *Chapters 2 and 3.*

In the following, the case when $f(A,s) = e^{As}$ will be particularly important. For $A \in M_n(\mathbb{R})$, the matrix exponential function e^{As} is defined using the Taylor series expansion as follows

$$e^{As} = \sum_{k=0}^{\infty} \frac{(As)^k}{k!}.$$

Note that e^{As} is again a matrix. It can be shown that $\frac{d}{ds}e^{As} = Ae^{As}$, which will also be useful later.

In general, a matrix function can be defined using the Jordan decomposition of a matrix, which can be understood as a generalization of a matrix diagonalization. Recall that any matrix $A \in M_n(\mathbb{C})$ can be uniquely expressed in the form $A = PJP^{-1}$, where P is a regular matrix and J is a Jordan matrix, i.e. a block-diagonal matrix with Jordan blocks on its diagonal. Each Jordan block is an upper triangular matrix with the same scalar on the main diagonal, it has entries 1 above the main diagonal, and it can be decomposed into the sum of a diagonal matrix and a nilpotent matrix, that is, a matrix for which $N^k = 0$, where k is the dimension of the Jordan block. The matrix function is then defined as $f(A) = Pf(J)P^{-1}$, so e^{As} can be defined as $e^{As} = Pe^{Js}P^{-1}$. More details about Jordan matrices and Jordan decomposition can be found in [15], Chapter 3.

5.1 Definition

The following theorem provides the definition of a multivariate supOU process, as well as sufficient conditions that ensure its existence.

Theorem 4 (see [6, Theorem 3.1]). *Let* Λ *be an* $\mathbb{R}^d$*-valued Lévy basis on* $M_d^- \times \mathbb{R}$ *with generating quadruple* $(\mathbf{a}, \Sigma, \mu, \pi)$ *satisfying*

$$\int_{\|\mathbf{x}\|>1} \log(\|\mathbf{x}\|)\, \mu(d\mathbf{x}) < \infty, \tag{5.2}$$

and assume there exist measurable functions $\rho : M_d^- \to \mathbb{R}_+\backslash\{0\}$ *and* $\eta : M_d^- \to [1,\infty)$ *such that*

$$\|e^{As}\| \leq \eta(A)e^{-\rho(A)s}, \quad \forall s \in \mathbb{R}_+,\ \pi\text{-a.s.}, \tag{5.3}$$

and

$$\int_{M_d^-} \frac{\eta(A)^2}{\rho(A)}\, \pi(dA) < \infty. \tag{5.4}$$

A d-dimensional supOU process $X = \{X(t),\, t \in \mathbb{R}\}$ given by

$$X(t) = \int_{M_d^-} \int_{-\infty}^{t} e^{A(t-s)}\, \Lambda(dA, ds) \tag{5.5}$$

is well-defined for all $t \in \mathbb{R}$ and stationary. The distribution of $X(t)$ is infinitely divisible with characteristic triplet $(\mathbf{a}_X, \Sigma_X, \mu_X)$ given by

$$\mathbf{a}_X = \int_{M_d^-} \int_{\mathbb{R}_+} \left(e^{As}\mathbf{a} + \int_{\mathbb{R}^d} e^{As}\mathbf{x} \left(\mathbf{1}_{[0,1]} \left(\|e^{As}\mathbf{x}\| \right) - \mathbf{1}_{[0,1]}(\|\mathbf{x}\|) \right) \mu(d\mathbf{x}) \right) ds\, \pi(dA),$$
$$\Sigma_X = \int_{M_d^-} \int_{\mathbb{R}_+} e^{As}\Sigma e^{A^T s}\, ds\, \pi(dA),$$
$$\mu_X(B) = \int_{M_d^-} \int_{\mathbb{R}_+} \int_{\mathbb{R}^d} \mathbf{1}_B \left(e^{As}\mathbf{x} \right) \mu(d\mathbf{x})\, ds\, \pi(dA).$$

for all Borel sets $B \subseteq \mathbb{R}^d$.

Proof. It follows from Proposition 8 that the necessary and sufficient conditions for the existence of integral are

$$\int_{M_d^-} \int_{\mathbb{R}_+} \left\| e^{As}\mathbf{a} + \int_{\mathbb{R}^d} e^{As}\mathbf{x} \left(\mathbf{1}_{[0,1]}(\|e^{As}\mathbf{x}\|) - \mathbf{1}_{[0,1]}(\|\mathbf{x}\|) \right) \mu(d\mathbf{x}) \right\| ds\, \pi(dA) < \infty, \tag{5.6}$$

$$\int_{M_d^-} \int_{\mathbb{R}_+} \left\| e^{As}\Sigma e^{A^T s} \right\| ds\, \pi(dA) < \infty, \tag{5.7}$$

$$\int_{M_d^-} \int_{\mathbb{R}_+} \int_{\mathbb{R}^d} \left(1 \wedge \|e^{As}\mathbf{x}\|^2 \right) \mu(d\mathbf{x})\, ds\, \pi(dA) < \infty. \tag{5.8}$$

We will first show that (5.8) holds. From condition (5.3), it follows that

$$\begin{aligned} &\int_{M_d^-} \int_{\mathbb{R}_+} \int_{\mathbb{R}^d} \left(1 \wedge \|e^{As}\mathbf{x}\|^2 \right) \mu(d\mathbf{x})\, ds\, \pi(dA) \\ &\quad \leq \int_{M_d^-} \int_{\mathbb{R}_+} \int_{\mathbb{R}^d} \left(1 \wedge \eta(A)^2 e^{-2\rho(A)s} \|\mathbf{x}\|^2 \right) \mu(d\mathbf{x})\, ds\, \pi(dA). \end{aligned}$$

We can calculate the last integral by splitting it into two cases, depending on whether the condition

$$\eta(A)^2 e^{-2\rho(A)s} \|\mathbf{x}\|^2 > 1 \tag{5.9}$$

or

$$\eta(A)^2 e^{-2\rho(A)s} \|\mathbf{x}\|^2 \leq 1 \tag{5.10}$$

holds.

Since $e^{-2\rho(A)s} < 1$, condition (5.9) can only hold if $\eta(A)^2\|\mathbf{x}\|^2 > 1$, that is $\|\mathbf{x}\| > 1/\eta(A)$, with an additional constraint on s. We have

$$\eta(A)^2 e^{-2\rho(A)s} \|\mathbf{x}\|^2 > 1 \iff s < \frac{\log(\eta(A)\|\mathbf{x}\|)}{\rho(A)}.$$

Condition (5.10) can hold for both $\eta(A)^2\|\mathbf{x}\|^2 \le 1$ and $\eta(A)^2\|\mathbf{x}\|^2 > 1$. In the first case, i.e. for $\|\mathbf{x}\| \le 1/\eta(A)$, (5.10) holds for every $s \in \mathbb{R}_+$. In the second case, i.e. for $\|\mathbf{x}\| > 1/\eta(A)$, (5.10) holds when $s \ge \frac{\log(\eta(A)\|\mathbf{x}\|)}{\rho(A)}$.

We obtain

$$
\begin{aligned}
&\int_{M_d^-}\int_{\mathbb{R}_+}\int_{\mathbb{R}^d}\left(1\wedge \eta(A)^2 e^{-2\rho(A)s}\|\mathbf{x}\|^2\right)\mu(d\mathbf{x})\,ds\,\pi(dA)\\
&= \int_{M_d^-}\int_0^{\frac{\log(\eta(A)\|\mathbf{x}\|)}{\rho(A)}}\int_{\|\mathbf{x}\|>1/\eta(A)}\mu(d\mathbf{x})\,ds\,\pi(dA)\\
&\quad+\int_{M_d^-}\int_{\mathbb{R}_+}\int_{\|\mathbf{x}\|\le 1/\eta(A)}\left(\eta(A)^2 e^{-2\rho(A)s}\|\mathbf{x}\|^2\right)\mu(d\mathbf{x})\,ds\,\pi(dA)\\
&\quad+\int_{M_d^-}\int_{\frac{\log(\eta(A)\|\mathbf{x}\|)}{\rho(A)}}^{+\infty}\int_{\|\mathbf{x}\|>1/\eta(A)}\left(\eta(A)^2 e^{-2\rho(A)s}\|\mathbf{x}\|^2\right)\mu(d\mathbf{x})\,ds\,\pi(dA)\\
&\overset{(*)}{=}\int_{M_d^-}\int_{\|\mathbf{x}\|>1/\eta(A)}\frac{\log(\eta(A)\|\mathbf{x}\|)}{\rho(A)}\mu(d\mathbf{x})\,\pi(dA)\\
&\quad+\int_{M_d^-}\int_{\|\mathbf{x}\|\le 1/\eta(A)}\frac{\eta(A)^2\|\mathbf{x}\|^2}{2\rho(A)}\mu(d\mathbf{x})\,\pi(dA)\\
&\quad+\int_{M_d^-}\int_{\|\mathbf{x}\|>1/\eta(A)}\frac{1}{2\rho(A)}\mu(d\mathbf{x})\,\pi(dA)\\
&=\int_{M_d^-}\int_{\|\mathbf{x}\|>1/\eta(A)}\frac{\log(\eta(A)\|\mathbf{x}\|)+1/2}{\rho(A)}\mu(d\mathbf{x})\,\pi(dA)\\
&\quad+\int_{M_d^-}\int_{\|\mathbf{x}\|\le 1/\eta(A)}\frac{\eta(A)^2\|\mathbf{x}\|^2}{2\rho(A)}\mu(d\mathbf{x})\,\pi(dA),
\end{aligned}
$$

where equality $(*)$ follows from

$$\int_{\mathbb{R}_+} e^{-2\rho(A)s}\,ds = \frac{1}{2\rho(A)}$$

and

$$\int_{\frac{\log(\eta(A)\|\mathbf{x}\|)}{\rho(A)}}^{+\infty} e^{-2\rho(A)s}\,ds = \frac{1}{2\rho(A)\eta(A)^2\|\mathbf{x}\|^2}.$$

The finiteness of the second integral in the last expression follows from (5.4), $\eta(A) \ge 1$, and the fact that μ is a Lévy measure, so it satisfies

$$\int_{\|\mathbf{x}\|\le 1}\|\mathbf{x}\|^2\,\mu(d\mathbf{x}) < \infty.$$

To ensure the finiteness of the first integral, we have

$$
\begin{aligned}
&\int_{M_d^-}\int_{\|\mathbf{x}\|>1/\eta(A)} \frac{\log(\eta(A)\|\mathbf{x}\|)+1/2}{\rho(A)}\,\mu(d\mathbf{x})\,\pi(dA)\\
&=\int_{M_d^-}\int_{\|\mathbf{x}\|>1} \frac{\log(\eta(A)\|\mathbf{x}\|)+1/2}{\rho(A)}\,\mu(d\mathbf{x})\,\pi(dA)\\
&\quad+\int_{M_d^-}\int_{1/\eta(A)<\|\mathbf{x}\|\le 1} \frac{\log(\eta(A)\|\mathbf{x}\|)+1/2}{\rho(A)}\,\mu(d\mathbf{x})\,\pi(dA)\\
&\le\int_{M_d^-}\int_{\|\mathbf{x}\|>1} \frac{\log(\eta(A))+\log(\|\mathbf{x}\|)+1/2}{\rho(A)}\,\mu(d\mathbf{x})\,\pi(dA)\\
&\quad+\int_{M_d^-}\int_{\|\mathbf{x}\|\le 1} \frac{\eta(A)^2\|\mathbf{x}\|^2}{\rho(A)}\,\mu(d\mathbf{x})\,\pi(dA)\\
&=\int_{M_d^-}\frac{\log(\eta(A))}{\rho(A)}\pi(dA)\int_{\|\mathbf{x}\|>1}\mu(d\mathbf{x})\\
&\quad+\int_{M_d^-}\frac{\eta(A)^2}{\rho(A)}\pi(dA)\int_{\|\mathbf{x}\|\le 1}\|\mathbf{x}\|^2\,\mu(d\mathbf{x})\\
&\quad+\int_{M_d^-}\frac{1}{\rho(A)}\pi(dA)\int_{\|\mathbf{x}\|>1}(\log(\|\mathbf{x}\|)+1/2)\,\mu(d\mathbf{x}),
\end{aligned}
$$

so the finiteness follows from (5.2), (5.4), $\eta(A)\ge 1$ and μ being a Lévy measure.

Using (5.4), condition (5.7) follows from

$$
\begin{aligned}
\int_{M_d^-}\int_{\mathbb{R}_+}\left\|e^{As}\Sigma e^{A^Ts}\right\|\,ds\,\pi(dA) &\le \|\Sigma\|\int_{M_d^-}\int_{\mathbb{R}_+}\eta(A)^2e^{-2\rho(A)s}\,ds\,\pi(dA)\\
&=\|\Sigma\|\int_{M_d^-}\frac{\eta(A)^2}{2\rho(A)}\,\pi(dA).
\end{aligned}
$$

To show (5.6) we first have

$$
\begin{aligned}
&\int_{M_d^-}\int_{\mathbb{R}_+}\left\|e^{As}\mathbf{a}+\int_{\mathbb{R}^d}e^{As}\mathbf{x}\left(\mathbf{1}_{[0,1]}(\|e^{As}\mathbf{x}\|)-\mathbf{1}_{[0,1]}(\|\mathbf{x}\|)\right)\mu(d\mathbf{x})\right\|\,ds\,\pi(dA)\\
&\le\int_{M_d^-}\int_{\mathbb{R}_+}\|e^{As}\mathbf{a}\|\,ds\,\pi(dA)\\
&\quad+\int_{M_d^-}\int_{\mathbb{R}_+}\left\|\int_{\mathbb{R}^d}e^{As}\mathbf{x}\left(\mathbf{1}_{[0,1]}\left(\|e^{As}\mathbf{x}\|\right)-\mathbf{1}_{[0,1]}(\|\mathbf{x}\|)\right)\mu(d\mathbf{x})\right\|\,ds\,\pi(dA)
\end{aligned}
$$

Again by using condition (5.4), the finiteness of the first integral in the previous expression follows from

$$
\begin{aligned}
\int_{M_d^-}\int_{\mathbb{R}_+}\|e^{As}\mathbf{a}\|\,ds\,\pi(dA) &\le \|\mathbf{a}\|\int_{M_d^-}\int_{\mathbb{R}_+}\eta(A)e^{-\rho(A)s}\,ds\,\pi(dA)\\
&=\|\mathbf{a}\|\int_{M_d^-}\frac{\eta(A)}{\rho(A)}\,\pi(dA)<\infty.
\end{aligned}
$$

The finiteness of the second integral follows from

$$\begin{aligned}
&\int_{M_d^-}\int_{\mathbb{R}_+}\left\|\int_{\mathbb{R}^d} e^{As}\mathbf{x}\left(\mathbf{1}_{[0,1]}\left(\|e^{As}\mathbf{x}\|\right)-\mathbf{1}_{[0,1]}(\|\mathbf{x}\|)\right)\mu(d\mathbf{x})\right\| ds\,\pi(dA)\\
&\leq \int_{M_d^-}\int_{\mathbb{R}_+}\int_{\|\mathbf{x}\|\leq 1,\|e^{As}\mathbf{x}\|\geq 1}\|e^{As}\mathbf{x}\|\,\mu(d\mathbf{x})\,ds\,\pi(dA)\\
&\quad+\int_{M_d^-}\int_{\mathbb{R}_+}\int_{\|\mathbf{x}\|\geq 1,\|e^{As}\mathbf{x}\|\leq 1}\|e^{As}\mathbf{x}\|\,\mu(d\mathbf{x})\,ds\,\pi(dA)\\
&\leq \int_{M_d^-}\int_{\mathbb{R}_+}\int_{\|\mathbf{x}\|\leq 1,\|e^{As}\mathbf{x}\|\geq 1}\|e^{As}\mathbf{x}\|^2\,\mu(d\mathbf{x})\,ds\,\pi(dA)\\
&\quad+\int_{M_d^-}\int_{\mathbb{R}_+}\int_{1\leq\|\mathbf{x}\|<e^{\rho(A)s/2}}\|\mathbf{x}\|\eta(A)e^{-\rho(A)s}\,\mu(d\mathbf{x})\,ds\,\pi(dA)\\
&\quad+\int_{M_d^-}\int_{\mathbb{R}_+}\int_{\|\mathbf{x}\|\geq e^{\rho(A)s/2}}\mu(d\mathbf{x})\,ds\,\pi(dA)\\
&\leq \int_{\|\mathbf{x}\|\leq 1}\|\mathbf{x}\|^2\,\mu(d\mathbf{x})\int_{M_d^-}\frac{\eta(A)^2}{2\rho(A)}\,\pi(dA)+\int_{M_d^-}\frac{2\eta(A)}{\rho(A)}\,\pi(dA)\int_{\|\mathbf{x}\|>1}\mu(d\mathbf{x})\\
&\quad+\int_{M_d^-}\frac{2}{\rho(A)}\,\pi(dA)\int_{\|\mathbf{x}\|>1}\log(\|\mathbf{x}\|)\,\mu(d\mathbf{x}),
\end{aligned}$$

by using (5.2), (5.4) and the fact that μ is a Lévy measure. After establishing well-definedness, stationarity is evident. The fact that the distribution of $X(t)$ is infinitely divisible with characteristic triplet $(\mathbf{a}_X, \Sigma_X, \mu_X)$ follows from Proposition 8. □

The conditions (5.6), (5.7), and (5.8) from the proof of Theorem 4 are necessary and sufficient for the existence of the integral, but as they are difficult to verify, we replace them by the sufficient conditions (5.2), (5.3) and (5.4) from Theorem 4. For practical applications, it is important to understand how close these sufficient conditions are to being necessary, as discussed in Proposition 3.3 in [6]. Also, it should be noted that the conditions (5.2), (5.3) and (5.4) do not require integration with respect to both μ and π, but only with respect to one of them, which is not the case with the mentioned necessary and sufficient conditions.

This definition is consistent with the definition of the univariate supOU process introduced in Chapter 4. Specifically, for $d = 1$, we have $M_d^- = \mathbb{R}_-$, condition (5.2) is equivalent to condition (4.5), and for $\eta(\xi) = 1$ and $\rho(\xi) = -\xi$, $\xi \in \mathbb{R}_-$, condition (5.4) corresponds exactly to (4.6). In this case, condition (5.3) is trivially satisfied.

Example 7 illustrates how κ and ρ can be specified in a measurable way, and Example 8 is a generalization of Example 5. It defines a supOU process for which we will later explicitly calculate second-order moments.

Example 7 (see [6, Example 3.5])**.** *Let*

$$\begin{aligned}\mathscr{D}_d^- &:= \{X \in M_d(\mathbb{R}) : X \textit{ is diagonal; all diagonal elements are strictly negative,} \\ &\qquad\qquad \textit{pairwise distinct and ordered such that } x_{ii} < x_{jj},\ \forall 1 \le i \le j \le d\}, \\ \mathscr{S}_d &:= \{X \in GL_d(\mathbb{R}) : \textit{ the first nonzero element in each column is } 1\}, \\ \mathscr{M}_d^- &:= \{SDS^{-1} : S \in \mathscr{S}_d, D \in \mathscr{D}_d^-\}.\end{aligned}$$

For a matrix $A = SDS^{-1} \in \mathscr{M}_d^-$, the matrix D is such that its diagonal elements are the eigenvalues of A, and the matrix S is such that its columns are the eigenvectors of A. In general, such a matrix decomposition is not unique, but it can be shown to be unique for $S \in \mathscr{S}_d, D \in \mathscr{D}_d^-$. We have that $e^{As} = Se^{Ds}S^{-1}$, from which it follows that

$$\|e^{As}\| \le \|S\|\|e^{Ds}\|\|S^{-1}\|,$$

so if we choose constants $\kappa = \|S\|\|S^{-1}\|$ and $\rho = -\max(\Re(\sigma(A)))$, where $\Re(\sigma(A))$ denotes the set of all real parts of the eigenvalues of A, we obtain

$$\|e^{As}\| \le \kappa e^{-\rho s}.$$

Due to the uniqueness of the decomposition, the continuous mapping defined by

$$\mathfrak{M} : \mathscr{S}_d \times \mathscr{D}_d^- \to \mathscr{M}_d^-,\ (S, D) \mapsto SDS^{-1}$$

is a bijection. Let's denote the inverse mapping by $\mathfrak{M}^{-1} = (\mathfrak{S}, \mathfrak{D})$. All of these mappings are measurable because the procedures used to compute the matrices S and D are measurable. If we define

$$\kappa : \mathscr{M}_d^- \to [1, \infty),\ \kappa(A) = \|\mathfrak{S}(A)\|\|(\mathfrak{S}(A))^{-1}\|$$

and

$$\rho : \mathscr{M}_d^- \to \mathbb{R}_+ \backslash \{0\},\ \rho(A) = -\max(\Re(\sigma(A))),$$

we can conclude that they are also measurable mappings on $\mathscr{M}_d^-$ and that the following holds

$$\|e^{As}\| \le \kappa(A) e^{-\rho(A)s}.$$

Using the κ and ρ defined in this way, we can define the probability measure π on $\mathscr{M}_d^-$, check whether the condition (5.4) holds, and thus define the supOU process.

Example 8. *Let Λ be a d-dimensional Lévy basis on $M_d^- \times \mathbb{R}$ with generating quadruple $(\mathbf{a}, \Sigma, \mu, \pi)$ satisfying*

$$\int_{\|\mathbf{x}\|>1} \log(\|\mathbf{x}\|)\, \mu(d\mathbf{x}) < \infty.$$

Let R be a random variable with $\Gamma(\alpha, \beta)$ distribution such that $\alpha > 1$ and $\beta > 0$, meaning R has the density

$$f(x) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x} \mathbf{1}_{[0,\infty)}(x),$$

and let B be a diagonalizable matrix in M_d^-. Define the probability measure π as the distribution of RB. It can be shown that the measure π defined in this way satisfies the conditions of Theorem 4, and that the process defined by

$$X(t) = \int_{M_d^-} \int_{-\infty}^{t} e^{A(t-s)} \Lambda(dA, ds)$$

exists and is stationary (see [6, Example 3.1]).

5.2 Moments

In this section, we will show under which conditions the supOU process has a finite r-th moment for $r \in (0, \infty)$ and provide expressions for $\mathbb{E}(X(0))$, $\mathrm{Var}(X(0))$ and $\mathrm{Cov}(X(h), X(0))$, i.e. the second-order moment structure.

Theorem 5 (see [6, Theorem 3.9]). *Let X be a stationary d-dimensional supOU process driven by a Lévy basis Λ satisfying the conditions (5.2), (5.3) and (5.4). It holds that*

(i) if $r \in (0, 2]$ and

$$\int_{\|\mathbf{x}\|>1} \|\mathbf{x}\|^r \mu(d\mathbf{x}) < \infty,$$

then X has a finite r-th moment, that is $\mathbb{E}(\|X(t)\|^r) < \infty$,

(ii) if $r \in (2, \infty)$ and

$$\int_{\|\mathbf{x}\|>1} \|\mathbf{x}\|^r \mu(d\mathbf{x}) < \infty, \quad \int_{M_d^-} \frac{\eta(A)^r}{\rho(A)} \pi(dA) < \infty,$$

then X has a finite r-th moment, that is $\mathbb{E}(\|X(t)\|^r) < \infty$.

Proof. Corollary 25.8 in [23] implies that it is sufficient to show

$$\int_{\|\mathbf{x}\|>1} \|\mathbf{x}\|^r \mu_X(d\mathbf{x}) < \infty.$$

We have

$$\begin{aligned}
\int_{\|\mathbf{x}\|>1} \|\mathbf{x}\|^r \mu_X(d\mathbf{x}) &= \int_{M_d^-} \int_{\mathbb{R}_+} \int_{\mathbb{R}^d} \|e^{As}\mathbf{x}\|^r \mathbf{1}_{(1,\infty)} \left(\|e^{As}\mathbf{x}\|\right) \mu(d\mathbf{x})\, ds\, \pi(dA) \\
&\le \int_{M_d^-} \int_{\mathbb{R}_+} \int_{\mathbb{R}^d} \eta(A)^r e^{-r\rho(A)s} \|\mathbf{x}\|^r \mathbf{1}_{(1,\infty)} \left(\eta(A) e^{-\rho(A)s} \|\mathbf{x}\|\right) \mu(d\mathbf{x})\, ds\, \pi(dA) \\
&= \int_{M_d^-} \int_{\|\mathbf{x}\|>1/\eta(A)} \int_0^{\log(\eta(A)\|\mathbf{x}\|)/\rho(A)} \eta(A)^r e^{-r\rho(A)s} \|\mathbf{x}\|^r\, ds\, \mu(d\mathbf{x})\, \pi(dA) \\
&= \int_{M_d^-} \int_{\|\mathbf{x}\|>1/\eta(A)} \frac{\eta(A)^r \|\mathbf{x}\|^r}{r\rho(A)} \left(1 - \frac{1}{\eta(A)^r \|\mathbf{x}\|^r}\right) \mu(d\mathbf{x})\, \pi(dA) \\
&= \int_{M_d^-} \int_{\|\mathbf{x}\|>1/\eta(A)} \frac{\eta(A)^r \|\mathbf{x}\|^r - 1}{r\rho(A)} \mu(d\mathbf{x})\, \pi(dA),
\end{aligned}$$

where the first equality follows from the expression for μ_X in Theorem 4. The finiteness of the expression

$$\int_{M_d^-}\int_{\|\mathbf{x}\|>1/\eta(A)} \frac{1}{r\rho(A)}\,\mu(d\mathbf{x})\,\pi(dA)$$

follows from the proof of Theorem 4, so it remains to show that

$$\int_{M_d^-}\int_{\|\mathbf{x}\|>1/\eta(A)} \frac{\eta(A)^r\|\mathbf{x}\|^r}{\rho(A)}\,\mu(d\mathbf{x})\,\pi(dA)$$

is finite, which follows from

$$\begin{aligned}
&\int_{M_d^-}\int_{\|\mathbf{x}\|>1/\eta(A)} \frac{\eta(A)^r\|\mathbf{x}\|^r}{\rho(A)}\,\mu(d\mathbf{x})\,\pi(dA)\\
&\leq \int_{M_d^-}\int_{\|\mathbf{x}\|>1} \frac{\eta(A)^r\|\mathbf{x}\|^r}{\rho(A)}\,\mu(d\mathbf{x})\,\pi(dA) + \int_{M_d^-}\int_{\|\mathbf{x}\|\leq 1} \frac{\eta(A)^{r\vee 2}\|\mathbf{x}\|^{r\vee 2}}{\rho(A)}\,\mu(d\mathbf{x})\,\pi(dA).
\end{aligned}$$

Using condition (5.4) and the fact that μ is a Lévy measure, (i) and (ii) follow. □

Theorem 6 (see [6, Theorem 3.11])**.** *Let X be a stationary d-dimensional supOU process driven by a Lévy basis Λ satisfying the conditions (5.2), (5.3) and (5.4). Assume additionally that $\int_{\mathbb{R}^d}\|\mathbf{x}\|^2\mu(d\mathbf{x})<\infty$. Then $\mathbb{E}(\|X_0\|^2)<\infty$ and*

$$\mathbb{E}\,(X(0)) = -\int_{M_d^-} A^{-1}\left(\mathbf{a}+\int_{\|\mathbf{x}\|>1}\mathbf{x}\,\mu(d\mathbf{x})\right)\pi(dA), \tag{5.11}$$

$$\mathrm{Var}\,(X(0)) = -\int_{M_d^-} (\mathscr{A}(A))^{-1}\left(\Sigma+\int_{\mathbb{R}^d}\mathbf{x}\mathbf{x}^T\,\mu(d\mathbf{x})\right)\pi(dA), \tag{5.12}$$

$$\mathrm{Cov}\,(X(\tau),X(0)) = -\int_{M_d^-} e^{A\tau}\,(\mathscr{A}(A))^{-1}\left(\Sigma+\int_{\mathbb{R}^d}\mathbf{x}\mathbf{x}^T\,\mu(d\mathbf{x})\right)\pi(dA), \tag{5.13}$$

for $\tau\in\mathbb{R}^+$ and $\mathscr{A}(A): M_d(\mathbb{R})\to M_d(\mathbb{R})$, $X \overset{\mathscr{A}(A)}{\mapsto} AX+XA^T$.

Moreover, it holds that

$$\lim_{\tau\to\infty}\mathrm{Cov}\,(X(\tau),X(0)) = 0. \tag{5.14}$$

Proof. Theorem 5 implies the finiteness of the second moments. Example 25.12 in [23] states that for every Lévy process $\{L(t), t\geq 0\}$ in $\mathbb{R}^d$ with characteristic triplet $(\mathbf{a},\Sigma,\mu)$, it holds that

$$\mathbb{E}(L(t)) = t\left(\mathbf{a}+\int_{\|\mathbf{x}\|>1}\mathbf{x}\,\mu(d\mathbf{x})\right). \tag{5.15}$$

Since $X(0)$ is an infinitely divisible random vector with characteristic triplet $(\mathbf{a}_X,\Sigma_X,\mu_X)$, from the previous expression it follows that

$$\mathbb{E}\,(X(0)) = \mathbb{E}(L_1(1)) = \mathbf{a}_X+\int_{\|\mathbf{x}\|>1}\mathbf{x}\,\mu_X(d\mathbf{x}),$$

where $\{L_1(t),\, t \geq 0\}$ is the corresponding d-dimesional Lévy process such that $X(0) \overset{d}{=} L_1(1)$. Now, using the expressions for $\mathbf{a}_X$ and μ_X from Theorem 4, we obtain

$$\mathbb{E}\left(X(0)\right) = \int_{M_d^-} \int_{\mathbb{R}^+} e^{As} \left(\mathbf{a} + \int_{\|\mathbf{x}\|>1} \mathbf{x}\, \mu(d\mathbf{x}) \right) ds\, \pi(dA).$$

Note that $e^{As} = \frac{d}{ds} A^{-1} e^{As}$. By substituting this expression into the above integral and integrating with respect to s, we obtain (5.11).

In a similar way, using Example 25.12 in [23], we obtain

$$\begin{aligned}
\mathrm{Var}\left(X(0)\right) &= \Sigma_X + \int_{\mathbb{R}^d} \mathbf{x}\mathbf{x}^T\, \mu_X(d\mathbf{x}) \\
&= \int_{M_d^-} \int_{\mathbb{R}_+} e^{As} \left(\Sigma + \int_{\mathbb{R}^d} \mathbf{x}\mathbf{x}^T\, \mu(d\mathbf{x}) \right) e^{A^T s}\, ds\, \pi(dA)
\end{aligned}$$

Now, by integrating over s, we obtain (5.12).

Furthermore, it holds

$$\begin{aligned}
\mathrm{Cov}\left(X(\tau), X(0)\right) &= \mathrm{Cov}\left(\int_{M_d^-} \int_{-\infty}^{\tau} e^{A(\tau-s)}\, \Lambda(dA, ds), \int_{M_d^-} \int_{-\infty}^{0} e^{-As}\, \Lambda(dA, ds) \right) \\
&= \mathrm{Cov}\left(\int_{M_d^-} \int_{-\infty}^{0} e^{A(\tau-s)}\, \Lambda(dA, ds), \int_{M_d^-} \int_{-\infty}^{0} e^{-As}\, \Lambda(dA, ds) \right) \\
&= \int_{M_d^-} e^{A\tau} \left(\int_{-\infty}^{0} e^{-As} \left(\Sigma + \int_{\mathbb{R}^d} \mathbf{x}\mathbf{x}^T\, \mu(d\mathbf{x}) \right) e^{-A^T s}\, ds \right) \pi(dA) \\
&= - \int_{M_d^-} e^{A\tau} (\mathscr{A}(A))^{-1} \left(\Sigma + \int_{\mathbb{R}^d} \mathbf{x}\mathbf{x}^T\, \mu(d\mathbf{x}) \right) \pi(dA),
\end{aligned}$$

where the second equality follows from the fact that Λ is a Lévy basis, so the random measures Λ on $M_d^- \times (0, \tau]$ and on $M_d^- \times (-\infty, 0]$ are independent.

To show that (5.14) holds, observe that

$$\begin{aligned}
&\left\| e^{A\tau} \left(\int_{-\infty}^{0} e^{-As} \left(\Sigma + \int_{\mathbb{R}^d} \mathbf{x}\mathbf{x}^T \mu(d\mathbf{x}) \right) e^{-A^T s} ds \right) \right\| \\
&\leq \int_{-\infty}^{0} \eta(A)^2 e^{\rho(A)(2s-\tau)}\, ds \left\| \Sigma + \int_{\mathbb{R}^d} \mathbf{x}\mathbf{x}^T \mu(d\mathbf{x}) \right\| \\
&\leq \frac{\eta(A)^2}{2\rho(A)} \left\| \Sigma + \int_{\mathbb{R}^d} \mathbf{x}\mathbf{x}^T \mu(d\mathbf{x}) \right\| < \infty.
\end{aligned}$$

Since $\lim_{\tau \to \infty} e^{A\tau} = 0$ for all $A \in M_d^-$, by the dominated convergence theorem, the statement follows. $\square$

Note that the expression $\mathbf{a} + \int_{\|\mathbf{x}\|>1} \mathbf{x}\, \mu(d\mathbf{x})$ in (5.11) corresponds to the expectation of the underlying Lévy process L, which follows from (5.15). For $d = 1$, we have

$$\mathbb{E}\left(X(0)\right) = -\mathbb{E}\left(L(1)\right) \int_{\mathbb{R}_-} \xi^{-1} \pi(d\xi) = \rho\, \mathbb{E}\left(L(1)\right).$$

In the one-dimensional case, for the mapping $\mathscr{A}(A) : M_d(\mathbb{R}) \to M_d(\mathbb{R})$, $X \overset{\mathscr{A}(A)}{\mapsto} AX + XA^T$, the corresponding mapping is defined by $\mathscr{A}(\xi) : \mathbb{R}_- \to \mathbb{R}_-$, $x \overset{\mathscr{A}(\xi)}{\mapsto} 2x\xi$, and the inverse mapping is given by $x \overset{\mathscr{A}^{-1}(\xi)}{\mapsto} \frac{x}{2\xi}$. Also, the expression $\Sigma + \int_{\mathbb{R}^d} \mathbf{x}\mathbf{x}^T \mu(d\mathbf{x})$ in (5.12) corresponds to the variance of the underlying Lévy process, so we obtain

$$\operatorname{Var}(X(0)) = -\int_{\mathbb{R}_-} \frac{\operatorname{Var}(L(1))}{2\xi} \pi(d\xi) = \frac{\rho}{2} \operatorname{Var}(L(1)),$$

which corresponds to (4.12). Similarly, for the covariance, we obtain

$$\operatorname{Cov}(X(\tau), X(0)) = -\int_{\mathbb{R}_-} e^{\xi\tau} \frac{\operatorname{Var}(L(1))}{2\xi} \pi(d\xi) = -\frac{\operatorname{Var}(L(1))}{2} \int_{\mathbb{R}_-} \xi^{-1} e^{\xi\tau} \pi(d\xi),$$

which corresponds to (4.11).

Property (5.14) is not obvious and requires a detailed proof because, even in the multivariate case, expression (5.13) does not show that the covariance function always decays exponentially. It is also possible for the multivariate supOU processes to exhibit long-range dependence. By long-range dependence in this context, we mean that at least one element of the covariance function asymptotically behaves like $\tau^{-\alpha}$, where $\tau \to \infty$ and $\alpha \in (0, 1)$.

Example 9. *Assume that Λ and π are defined as in Example 8, and additionally, suppose that*

$$\int_{\mathbb{R}^d} \|\mathbf{x}\|^2 \mu(d\mathbf{x}) < \infty$$

holds, which ensures the existence of second-order moments. We have

$$\begin{aligned}
\mathbb{E}(X(0)) &= -\int_{M_d^-} A^{-1} \left(\mathbf{a} + \int_{\|\mathbf{x}\|>1} \mathbf{x}\, \mu(d\mathbf{x}) \right) \pi(dA) \\
&= -\int_{\mathbb{R}_+} (Br)^{-1} \left(\mathbf{a} + \int_{\|\mathbf{x}\|>1} \mathbf{x}\, \mu(d\mathbf{x}) \right) \frac{\beta^\alpha}{\Gamma(\alpha)} r^{\alpha-1} e^{-\beta r}\, dr \\
&= -\frac{\beta^\alpha}{\Gamma(\alpha)} \int_{\mathbb{R}_+} e^{-\beta r} r^{\alpha-2}\, dr\, B^{-1} \left(\mathbf{a} + \int_{\|\mathbf{x}\|>1} \mathbf{x}\, \mu(d\mathbf{x}) \right) \\
&= -\frac{\beta^\alpha}{\Gamma(\alpha)} \Gamma(\alpha-1) \beta^{1-\alpha} B^{-1} \left(\mathbf{a} + \int_{\|\mathbf{x}\|>1} \mathbf{x}\, \mu(d\mathbf{x}) \right) \\
&= -\frac{\beta}{\alpha-1} B^{-1} \left(\mathbf{a} + \int_{\|\mathbf{x}\|>1} \mathbf{x}\, \mu(d\mathbf{x}) \right),
\end{aligned}$$

where we use the fact that $\int_0^\infty r^{\alpha-1} e^{-\beta r}\, dr = \Gamma(\alpha)\beta^{-\alpha}$ and $\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$.

Similarly, for the variance we obtain

$$\begin{aligned}\mathrm{Var}\,(X(0)) &= -\int_{M_d^-} (\mathscr{A}(A))^{-1}\left(\Sigma + \int_{\mathbb{R}^d} \mathbf{x}\mathbf{x}^T\,\mu(d\mathbf{x})\right)\pi(dA)\\ &= -\int_{\mathbb{R}_+} (\mathscr{A}(Br))^{-1}\left(\Sigma + \int_{\mathbb{R}^d} \mathbf{x}\mathbf{x}^T\,\mu(d\mathbf{x})\right)\frac{\beta^\alpha}{\Gamma(\alpha)} r^{\alpha-1}e^{-\beta r}\,dr\\ &= -\frac{\beta^\alpha}{\Gamma(\alpha)}\int_{\mathbb{R}_+} e^{-\beta r} r^{\alpha-2}\,dr\,\mathscr{B}^{-1}\left(\Sigma + \int_{\mathbb{R}^d} \mathbf{x}\mathbf{x}^T\,\mu(d\mathbf{x})\right)\\ &= -\frac{\beta}{\alpha-1}\mathscr{B}^{-1}\left(\Sigma + \int_{\mathbb{R}^d} \mathbf{x}\mathbf{x}^T\,\mu(d\mathbf{x})\right),\end{aligned}$$

where $\mathscr{B} : M_d(\mathbb{R}) \to M_d(\mathbb{R}),\ X \overset{\mathscr{B}}{\mapsto} BX + XB^T$.

From (5.13) *it follows that*

$$\begin{aligned}\mathrm{Cov}\,(X(\tau), X(0)) &= -\int_{M_d^-} e^{A\tau}(\mathscr{A}(A))^{-1}\left(\Sigma + \int_{\mathbb{R}^d} \mathbf{x}\mathbf{x}^T\,\mu(d\mathbf{x})\right)\pi(dA)\\ &= -\int_{\mathbb{R}_+} e^{Br\tau}(\mathscr{A}(Br))^{-1}\left(\Sigma + \int_{\mathbb{R}^d} \mathbf{x}\mathbf{x}^T\,\mu(d\mathbf{x})\right)\frac{\beta^\alpha}{\Gamma(\alpha)} r^{\alpha-1}e^{-\beta r}dr\\ &= -\frac{\beta^\alpha}{\Gamma(\alpha)}\int_{\mathbb{R}_+} e^{Br\tau-\beta I_d r} r^{\alpha-2}\,dr\left(\mathscr{B}^{-1}\left(\Sigma + \int_{\mathbb{R}^d} \mathbf{x}\mathbf{x}^T\,\mu(d\mathbf{x})\right)\right).\end{aligned}$$

Since B *is assumed to be diagonalizable, we know that there exists a matrix* $U \in GL_d(\mathbb{C})$ *and scalars* $\lambda_1, \lambda_2, \ldots, \lambda_d \in (-\infty, 0) + i\mathbb{R}$ *such that*

$$UBU^{-1} = \begin{bmatrix}\lambda_1 & 0 & \cdots & 0\\ 0 & \lambda_2 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & \lambda_d\end{bmatrix}.$$

Using the previous conclusion and the fact that $\int_0^\infty r^{z-1}e^{-kr}\,dr = \Gamma(z)k^{-z}$ *also holds for all* $z, k \in (0,\infty) + i\mathbb{R}$*, we obtain*

$$\begin{aligned}&\int_{\mathbb{R}_+} e^{Br\tau-\beta I_d r} r^{\alpha-2}dr = \int_{R_+} e^{-r(\beta I_d - B\tau)} r^{\alpha-2}dr\\ &= U\int_{\mathbb{R}_+} \exp\left(-r\left(\beta I_d - \begin{bmatrix}\lambda_1 & 0 & \cdots & 0\\ 0 & \lambda_2 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & \lambda_d\end{bmatrix}\tau\right)\right) r^{\alpha-2}\,dr\,U^{-1}\\ &= \Gamma(\alpha-1)U\left(\beta I_d - \begin{bmatrix}\lambda_1 & 0 & \cdots & 0\\ 0 & \lambda_2 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & \lambda_d\end{bmatrix}\tau\right)^{1-\alpha} U^{-1}\\ &= \Gamma(\alpha-1)(\beta I_d - B\tau)^{1-\alpha}.\end{aligned}$$

Thus, we have

$$\begin{aligned}\operatorname{Cov}(X(\tau), X(0)) &= -\frac{\beta^\alpha}{\Gamma(\alpha)}\Gamma(\alpha-1)(\beta I_d - B\tau)^{1-\alpha}\mathscr{B}^{-1}\left(\Sigma + \int_{\mathbb{R}^d} \mathbf{x}\mathbf{x}^T\, \mu(d\mathbf{x})\right)\\ &= -\frac{\beta^\alpha}{\alpha-1}(\beta I_d - B\tau)^{1-\alpha}\mathscr{B}^{-1}\left(\Sigma + \int_{\mathbb{R}^d} \mathbf{x}\mathbf{x}^T\, \mu(d\mathbf{x})\right).\end{aligned}$$

We can see that the covariance function decays polynomially, and for $\alpha \in (1,2)$, the property of long-range dependence is achieved.

Example 10 (see [6, Example 3.2]). *The previous example can be generalized to the case when π is defined as $\pi = \sum_{i=1}^m w_i\pi_i$, where $w_1, \dots, w_m \in [0,1]$ with $\sum_{i=1}^m w_i = 1$, and for $i \in \{1, \dots, m\}$ probability measure π_i is defined as the distribution of R_iB_i, where $R_i \sim \Gamma(\alpha_i, \beta_i)$, $\alpha_i > 1$, $\beta_i > 0$ and $B_1, \dots, B_m$ are diagonalizable matrices in M_d^-. In this case, we get*

$$\begin{aligned}\mathbb{E}(X(0)) &= -\sum_{i=1}^m \frac{w_i\beta_i}{\alpha_i - 1}B_i^{-1}\left(\mathbf{a} + \int_{\|\mathbf{x}\|>1} \mathbf{x}\, \mu(d\mathbf{x})\right),\\ \operatorname{Var}(X(0)) &= -\sum_{i=1}^m \frac{w_i\beta_i}{\alpha_i - 1}\mathscr{B}_i^{-1}\left(\Sigma + \int_{\mathbb{R}^d} \mathbf{x}\mathbf{x}^T\, \mu(d\mathbf{x})\right),\\ \operatorname{Cov}(X(\tau), X(0)) &= -\sum_{i=1}^m \frac{w_i\beta_i^{\alpha_i}}{\alpha_i - 1}(\beta_i I_d - B_i\tau)^{1-\alpha_i}\mathscr{B}_i^{-1}\left(\Sigma + \int_{\mathbb{R}^d} \mathbf{x}\mathbf{x}^T\, \mu(d\mathbf{x})\right),\end{aligned}$$

where $\mathscr{B}_i : M_d(\mathbb{R}) \to M_d(\mathbb{R})$, $X \overset{\mathscr{B}_i}{\mapsto} B_iX + XB_i^T$.

Example 11 (see [6, Example 3.4]). *Let $\mathbb{D}_2^{--}$ be the set of all 2×2 diagonal matrices with negative entries on the main diagonal and let Λ be a two-dimensional Lévy basis on $\mathbb{D}_2^{--} \times \mathbb{R}$ with generating quadruple $(\mathbf{a}, \Sigma, \mu, \pi)$ satisfying*

$$\int_{\mathbb{R}^2} (\|\mathbf{x}\|)^2\, \mu(d\mathbf{x}) < \infty.$$

Note that $\mathbb{D}_2^{--}$ can be identified with $(\mathbb{R}^{--})^2$, where $\mathbb{R}^{--}$ is the set of all negative real numbers (excluding 0). Let probability measure π on $\mathbb{D}_2^{--}$ be given by

$$\pi(dr_1, dr_2) = \frac{\beta_1^{\alpha_1}\beta_2^{\alpha_2}}{\Gamma(\alpha_1)\Gamma(\alpha_2)}(-r_1)^{\alpha_1-1}(-r_2)^{\alpha_2-1}e^{\beta_1r_1+\beta_2r_2}\mathbf{1}_{(\mathbb{R}^{--})^2}(r_1, r_2)\, dr_1\, dr_2,$$

where $\alpha_1, \alpha_2 > 1$ and $\beta_1, \beta_2 > 0$. We can observe that the diagonal elements are independent, and their absolute values follow gamma distributions. It can be shown that the conditions of Theorem 4 are satisfied, and that the stationary stochastic process defined by (5.5) with the given probability measure π is well-defined (see [6, Example 3.4]).

The process X is also two-dimensional, so let X_1 and X_2 denote its components. Let $P_1 : \mathbb{R}^2 \to \mathbb{R}$, $P((x_1, x_2)^T) = x_1$, be the projection onto the first coordinate. Define the Lévy *basis Λ_1 on $\mathbb{R}^{--} \times \mathbb{R}$ as $\Lambda_1(dr_1, ds) = P(\Lambda(P_1^{-1}(dr_1), ds)$ and the Lévy measure μ_1 on $\mathbb{R}$ as $\mu_1(dx_1) = \mu(P_1^{-1}(dx_1))$. Then Λ_1 has a generating quadruple $(a_1, \Sigma_{11}, \mu_1, \pi_1)$, where π_1 is defined with*

$$\pi_1(dr_1) = \frac{\beta_1^{\alpha_1}}{\Gamma(\alpha_1)} (-r_1)^{\alpha_1 - 1} e^{\beta_1 r_1} \mathbf{1}_{\mathbb{R}^{--}}(r_1)\, dr_1.$$

Also, it holds that

$$X_1(t) = \int_{\mathbb{R}^{--}} \int_{-\infty}^{t} e^{r_1(t-s)} \Lambda_1(dr_1, ds).$$

From (5.13) it follows that

$$\begin{aligned}
&\operatorname{Cov}(X_1(\tau), X_1(0)) \\
&= -\int_{\mathbb{R}^{--}} e^{r_1\tau} \frac{1}{2r_1} \left(\Sigma_{11} + \int_{\mathbb{R}} x^2 \mu_1(dx) \right) \frac{\beta_1^{\alpha_1}}{\Gamma(\alpha_1)} (-r_1)^{\alpha_1 - 1} e^{\beta r_1}\, dr_1 \\
&= \frac{\beta_1^{\alpha_1}}{2\Gamma(\alpha_1)} \int_{\mathbb{R}^{--}} e^{r_1(\beta_1 + \tau)} (-r_1)^{\alpha_1 - 2}\, dr_1 \left(\Sigma_{11} + \int_{\mathbb{R}} x^2 \mu_1(dx) \right) \\
&= \frac{\beta_1^{\alpha_1}}{2(\alpha_1 - 1)} (\beta_1 + \tau)^{1-\alpha_1} \left(\Sigma_{11} + \int_{\mathbb{R}} x^2 \mu_1(dx) \right).
\end{aligned}$$

Therefore, the correlation function for the process $\{X_1(t),\, t \in \mathbb{R}\}$ is given by

$$r(\tau) = \beta_1^{\alpha_1 - 1} (\beta_1 + \tau)^{1-\alpha_1}. \tag{5.16}$$

By applying an analogous procedure, we obtain the expression for the covariance and correlation functions of the process $\{X_2(t),\, t \in \mathbb{R}\}$. Thus, for $\alpha_1, \alpha_2 \in (1, 2)$, we have the property of long-range dependence in both components. Additionally, note that for $\alpha_1 = \alpha + 1$ and $\beta_1 = 1$, we obtain the same process and the same correlation function as described in Example 5.

The previous example is important because it demonstrates that we can define a two-dimensional supOU process with predetermined marginal distributions for the components X_1 and X_2. We know that the marginal distributions of the components are selfdecomposable distributions, given that these are one-dimensional processes. Therefore, instead of specifying μ, we can specify μ_1 and μ_2 to ensure that the marginal distributions of the components are the desired selfdecomposable distributions. If X has independent components, we can set $\mu(dx_1, dx_2) = \mu_1(dx_1) \times \delta_0(x_2) + \delta_0(x_1)\mu_2(dx_2)$, where δ_0 is the Dirac distribution with unit mass at 0.

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Summary

SupOU processes, or superpositions of Ornstein-Uhlenbeck type processes, belong to the class of stationary stochastic processes for which the marginal distribution and dependence structure can be modeled independently. To define them, the concepts of infinite divisibility and Lévy processes are first introduced. Their basic properties are given, and the connection between them is explained. The concept of selfdecomposable random variables is also defined, showing that they constitute the class of marginal distributions of Ornstein-Uhlenbeck type processes. Some basic properties of such processes are discussed, and the supOU process is characterized as an infinite superposition of Ornstein-Uhlenbeck type processes, obtained by randomizing the parameter using a probability measure. To make sense of the integral defined in this way, the concepts of a homogeneous infinitely divisible independently scattered random measure, or Lévy basis, and integration with respect to such a measure are introduced. The formal definition of the supOU process is then stated, followed by the derivation of the cumulant functions of the marginal distributions. The expression for the autocorrelation function of the supOU process is derived, showing that these processes can exhibit long-range dependence. The integrated process is briefly explained as an example important in applications. The definition of the supOU process is generalized to the multidimensional case, for which the conditions under which it has moments of any order are shown, along with expressions for the expectation, variance, and covariance of such a process.

Keywords

infinite divisibility, Lévy process, Ornstein-Uhlenbeck type process, Lévy basis, supOU process, long-range dependence

SupOU procesi

Sažetak

SupOU procesi, odnosno superpozicije procesa Ornstein-Uhlenbeckovog tipa, spadaju u klasu stacionarnih slučajnih procesa kojima se marginalna distribucija i struktura zavisnosti mogu modelirati nezavisno. Kako bismo ih definirali, najprije su uvedeni koncepti beskonačne djeljivosti i Lévyjevog procesa. Navedena su neka njihova osnovna svojstva te je objašnjena veza između njih. Definiran je i pojam selfdecomposable slučajnih varijabli, za koje se pokazuje da čine klasu marginalnih distribucija procesa Ornstein-Uhlenbeckovog tipa. Navedena su i neka osnovna svojstva takvih procesa, a supOU proces možemo shvatiti kao beskonačnu superpoziciju procesa Ornstein-Uhlenbeckovog tipa dobivenu randomiziranjem parametra pomoću vjerojatnosne mjere. Da bismo mogli dati smisao na taj način definiranom integralu, najprije je uveden pojam homogene beskonačno djeljive nezavisno raspršene slučajne mjere, odnosno Lévyjeve baze, te integracije u odnosu na takvu mjeru. Iskazana je formalna definicija supOU procesa, nakon koje su izvedeni izrazi za funkcije kumulanata marginalnih distribucija. Izveden je izraz za autokorelacijsku funkciju supOU procesa, iz kojeg se vidi da ovi procesi mogu imati svojstvo dugoročne zavisnosti. Ukratko je objašnjen integrirani proces, kao primjer važan u primjenama. Definicija supOU procesa poopćena je na višedimenzionalan slučaj, za koji je pokazano uz koje uvjete ima momente svakog reda te su dani izrazi za očekivanje, varijancu i kovarijancu ovakvog procesa.

Ključne riječi

beskonačna djeljivost, Lévijev proces, proces Ornstein-Uhlenbeckovog tipa, Lévijeva baza, supOU proces, dugoročna zavisnost

Biography

I was born on June 5, 2000, in Vinkovci. I attended Stjepan Radić Elementary School in Bok, after which I enrolled in the grammar school at Fra Martin Nedić School Center in Orašje. During my high school education, I participated in various competitions, including federal mathematics competitions. In 2019, I began my undergraduate studies in Mathematics at the Department of Mathematics (now School of Applied Mathematics and Informatics) in Osijek, which I completed in 2022 with the honor *summa cum laude*. My thesis, titled *Application of Cholesky Decomposition to Solving Linear Systems*, was supervised by Dr. Suzana Miodragović, Assistant Professor. That same year, I enrolled in the graduate program in Financial Mathematics and Statistics. During my undergraduate and graduate studies, I participated in preparing high school students for competitions and served as an undergraduate and graduate teaching assistant for the Mathematics Precourse and for the courses Linear Algebra 1, Linear Algebra 2, and Introduction to Probability and Statistics. I took part in the ECMI Modelling Week in 2023 and 2024. I am a recipient of the Lions Club Osijek Award for the most successful university students. During my studies, I also received three commendations for academic excellence, the Head of Department's Award, the Dean's Award, a commendation for the most successful undergraduate and graduate teaching assistants, and the Rector's Award for the seminar paper in the Mathematical Finance course.