On Consecutive Palindromes

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Summary

In this paper, we will try to characterize consecutive palindromes - positive integers simultaneously palindromic in multiple consecutive number bases. We search for solutions in the cases of two and three number bases and observe the fact that solutions likely do not exist in the case of four or more number bases. In practice, this requires solving systems of corresponding linear Diophantine equations.

This was motivated by the question “Can a positive integer be palindromic in more than three consecutive number bases?”. The question does not appear to have an obvious approach and remains open. The analysis of consecutive palindromes in three number bases suggests that the answer to the question is likely negative.

Key words

Palindromic numbers, number bases, linear Diophantine equations, linear Diophantine systems

O uzastopnim palindromima

Sažetak

U ovom radu pokušat ćemo karakterizirati uzastopne palindrome - brojeve koji su istovremeno palindromični u višestruko uzastopnih brojevnih baza. Tražimo rješenja u slučajevima dvije i tri brojevne baze, te promatramo činjenicu kako rješenja vjerojatno ne postoje u slučaju četiri ili više brojevnih baza. U praksi, ovo zahtjeva rješavanje sustava odgovarajućih linearnih Diofantskih jednadžbi.

Ovaj rad je motiviralo pitanje "Može li broj biti palindromičan u više od tri uzastopne brojevne baze?". Čini se kako pitanje nema očit pristup te ostaje otvoreno. Analiza uzastopnih palindroma u tri brojevne baze upućuje nas na to da bi odgovor na postavljeno pitanje bio negativan.

Ključne riječi

Palindromični brojevi, brojevne baze, linearne Diofantske jednadžbe, linearni Diofantski sustavi

Contents

1 Introduction

At the beginning of this chapter, we will introduce the idea of consecutive palindromes and present a few known problems related to palindromes. The key definitions will be laid out in the first section of the introduction chapter. In the second section of the introduction chapter, we mention a related problem to consecutive palindromes, the simultaneous palindromes. Beyond that, we start characterizing consecutive palindromes.

We say that a positive integer is palindromic (is a palindrome) in an integer number base $b \geq 2$, if it remains the same when its digits are reversed in that number base. Notice that all positive integers less than some number base b, have a one-digit representation in that number base, and hence are trivially palindromic in that number base. Based on this, we will exclude the one-digit numbers from the definition of consecutive palindromes.

A consecutive palindrome is a positive integer that is simultaneously palindromic in at least two consecutive number bases, having at least two digits in those number bases. For example, the number 10 is a consecutive palindrome, since it is palindromic in consecutive number bases 3 and 4, i.e.,

$$10 = 1 \cdot 3^2 + 0 \cdot 3^1 + 1 \cdot 3^0 = 2 \cdot 4^1 + 2 \cdot 4^0,$$

so its base three digits $(1, 0, 1)$ and base four digits $(2, 2)$ are the same when their order is reversed. The number 10 is also the smallest consecutive palindrome.

The results in this paper are founded on observations that we had previously discussed on Mathematics Stack Exchange (see [16, 17, 19]) and Math Overflow (see [18]), under username `Vepir`. Before we start, we would like to acknowledge a few unsolved problems related to palindromes, which initially sparked our interest in this topic. These were mainly observed in the decimal number base only, and are open for research.

- The problem of finding perfect powers $n^k, n > 1, k > 1$ that are palindromic was examined by Simmons in 1970 (see [15]). He obtains some results for $k = 2, 3, 4$, such as that there are infinitely many palindromic squares. He also conjectures that no solutions exist for $k > 4$.

- In the same paper, Simmons observes that the only known non-palindromic number whose cube is a palindrome is 2201 (see [15, Question 2]). He verifies this up to $2.8 \cdot 10^{14}$. This was later discussed in a book by Gardner in 1982 (see [7]), and later resurfaced on Math Stack Exchange in 2013 (see [13]). A similar claim does not necessarily hold in other number bases.

- Lychrel numbers, defined as positive integers that cannot form a palindrome through the iterative process of repeatedly reversing and adding their digits, are unsolved in the decimal number base. Candidates for such numbers are being collected in The On-Line Encyclopedia of Integer Sequences (OEIS), under the sequence A023108. However, when observing the Lychrel process in other number bases such as powers of two, the Lychrel numbers were successfully constructed (see [12]).

1.1 Number representations and palindromes

Regarding the notation used, we will remain in the realm of integers:

- $\mathbb{N} := \{1, 2, 3, \dots\}$, $\mathbb{N}_0 := \{0, 1, 2, 3, \dots\}$, $\mathbb{Z} := \{\dots, -2, -1, 0, 1, 2, \dots\}$.
- Let $n \leq m, (n, m) := \{n+1, n+2, \dots, m-1\} \subset \mathbb{Z}, [n, m] := \{n\} \cup (n, m) \cup \{m\}$, and similarly $[n, m) := \{n\} \cup (n, m)$ and $(n, m] := (n, m) \cup \{m\}$.

In this section, we present the key definitions and propositions for the number base representations and palindromes.

Definition 1.1. *Let $n, b \in \mathbb{N}$ and $b \geq 2$. If there exist integers $d \in \mathbb{N}, a_i \in [0, b), a_1 \neq 0, i \in [1, d]$ such that*

$$n = a_1 b^{d-1} + a_2 b^{d-2} + \cdots + a_{d-1} b + a_d = \sum_{i=1}^{d} a_i b^{d-i},$$

then we say that n has a d digit number representation in number base b, where a_i are called the digits of the number n in base b. a_1 is the leading digit, and a_d the unit digit. We write

$$n = (a_1, a_2, \dots, a_d)_b,$$

and call this the natural representation of number n in the number base b. We will refer to this as the normal form of the number n in base b.

It can be shown that a normal form of some n in some number base b is unique. This will follow from the uniqueness of remainders in the division theorem, also known as the Euclidean division.

Lemma 1.2 (see [14, Theorem 5. The Division Algorithm])**.** *Let $a, b \in \mathbb{Z}$, where $b > 0$. Then there exist unique integers q, r such that $a = bq + r$, where $r \in [0, b)$.*

Proof can be found in books or courses that cover introductory number theory (see [14]).

Proposition 1.3 (see [14, Theorem 6.])**.** *If $n, b \in \mathbb{N}, b \geq 2$, then the normal form of n in the number base b is unique.*

Proof:
We write the d digit normal form of some n in some base b:

$$n = a_d + a_{d-1} b + a_{d-2} b^2 \cdots + a_1 b^{d-1} = a_d + qb,$$

where $q = a_{d-1} b^0 + a_{d-2} b^1 + \cdots + a_1 b^{d-2}$. We know that $a_i \in [0, b)$. From Lemma 1.2, a_d is unique as it represents the remainder of division of n by b. Now, inductively observe that

$$\begin{aligned} q &= a_{d-1} + a_{d-2} b + a_{d-3} b^2 \cdots + a_1 b^{d-2} = a_{d-1} + q_{d-1} b \\ q_{d-1} &= a_{d-2} + a_{d-3} b + a_{d-4} b^2 \cdots + a_1 b^{d-3} = a_{d-2} + q_{d-2} b \\ &\vdots \\ q_3 &= a_2 + a_1 b = a_2 + q_2 b. \end{aligned}$$

Applying Lemma 1.2 we conclude that the remaining digits $a_{d-1}, \dots, a_2$ are uniquely determined as remainders of division by b. In the last step, a_1 is the unique quotient. $\square$

Notice that the proof of Proposition 1.3 gives us a way to write number n in base b.

Definition 1.4. *Let $b \geq 2$. We say that a positive integer $n \in \mathbb{N}$ is palindromic in the number base b, and call it a palindrome in the number base b if*

$$a_i = a_{d-i+1}, \quad \forall i \in [1, d],$$

where a_i are digits in the normal form of n in base b.

We say that n is nontrivially palindromic if $b \leq n-2$. Otherwise, if $n-1 \leq b$, then we say that n is trivially palindromic.

We say that n is strictly nonpalindromic if it is not a palindrome in any of the nontrivial number bases $2 \leq b \leq n-2$.

Trivial palindromes are, well, trivial. Notice that if $b = n-1$ and $n \geq 3$, we have a palindrome: $n = 1 \cdot (n-1) + 1 = (1,1)_{n-1}$. If $n = b$ we cannot have a palindrome: $n = 1 \cdot n + 0 = (1,0)_n$. Finally, if $n < b$, then $n = (n)_b$ is a one-digit palindrome.

Proposition 1.5 (see [22, Properties])**.** *If $n > 6$ is strictly nonpalindromic, then n is a prime number.*

Proof:
We will show that non-prime number $n > 6$ cannot be strictly nonpalindromic. If such n is not a prime number, then it must be composite and we can factorize it like $n = pq$. Let p be the smallest factor of n. Observe two cases:

- Assume that $p \neq q$.

  Because factors are multiples of primes, which are all odd except 2, we see that $q-p > 1$. The only time when $q-p \in \{0,1\}$ is possible, is if we have factors $2,3$ or $2,2$ or $p = q$, but this is impossible since $n > 6$ and $p \neq q$. That is, we have $p < q-1$.

  This means that n is palindromic in base $q-1$, since:

  $$n = pq = p(q-1) + p = (p,p)_{q-1}, p < q-1.$$

- Otherwise, if $p = q$, then $n = p^2$ is a square. If $p > 3$, then we have $2 < p-1$, and n is palindromic in number base $p-1$, since

  $$n = p^2 = (p-1)^2 + 2(p-1) + 1 = (1,2,1)_{p-1}, 2 < p-1.$$

  And specially, if $p = 3$, we have $9 = 1 \cdot 2^3 + 1 = (1,0,0,1)_2$, i.e., a binary palindrome. If $p = 2$, then $n = 4$, which contradicts with $n > 6$.

We have shown that composite number $n > 6$ has at least one nontrivial palindromic representation. This implies that only prime numbers and $\{1,4,6\}$ can be strictly nonpalindromic. $\square$

The converse of Proposition 1.5 does not hold. That is, if $n > 6$ is a prime, then n is not necessarily strictly nonpalindromic. For example, 13 is a prime, but is not strictly nonpalindromic, since $13 = (1, 1, 1)_3$ is a palindrome in base three.

Corollary 1.6. *If $n > 6$ is composite, then n has at least one nontrivial palindromic representation in some number base $2 \leq b \leq n - 2$.*

The corollary follows from the proof of Proposition 1.5.

1.2 Simultaneous and consecutive palindromes

For the convenience of the reader, in this section we state definitions concerning simultaneous and consecutive palindromes.

Firstly, we briefly introduce simultaneous palindromes and related results.

Definition 1.7. *A positive integer n is a simultaneous palindrome in number bases $b_1, b_2, \ldots$ if it is simultaneously palindromic in those number bases.*

For simultaneous palindromes, and palindromic additivity, research has been done in recent past. For example,

- In 2009, it was shown that there are exactly 203 numbers that are simultaneously palindromic in number bases 10 and some $b \neq 10$, and have $d \geq 2$ digits in both number bases. The result relies on exhaustive computation (see [8]).

- In 2010, Bašić shows that for any $K \in \mathbb{N}, d \geq 2$, there exists $n \in \mathbb{N}$ and a list of bases $\{b_1, b_2, \ldots, b_K\}$ such that n is a d digit palindrome in those bases (see [3]).

- Let $h < g$ be multiplicatively independent and $h|g$. In 2014, Berczes and Ziegler have shown for which such number bases h, g we have at most finitely many simultaneous palindromes. They also provided their upper bounds on the size of such palindromes. (see [4]).

- In 2016, it was shown that for a number base $g \geq 5$, every number can be written as a sum of three base g palindromes (see [6]).

- In 2018, Lo and Paz find all (base 10) positive integers a, b such that $b \pm a$ and ab are simultaneously palindromic (see [11]).

Sadly, the mentioned results do not produce significant conclusions about the consecutive palindromes themselves. Notice that consecutive palindromes are a special case of simultaneous palindromes:

Definition 1.8. *If n is nontrivially palindromic in $k \geq 2$ consecutive number bases $b, b+1, \ldots, b+k-1$, then we say that n is a consecutive palindrome. For $k = 2, 3, 4, \ldots,$ we call n a double, triple, quadruple,... palindrome, respectively.*

We say that a consecutive palindrome has d digits if it has d digits in the smallest consecutive palindromic base b. If it has d digits in all of the k consecutive palindromic bases, then we say that it is regular. Otherwise, it is an irregular consecutive palindrome.

For example, it can be shown that the smallest double palindrome is irregular and equals

$$10 = (1, 0, 1)_3 = (2, 2)_4.$$

Similarly, the smallest triple palindrome is regular and equals

$$178 = (4, 5, 4)_6 = (3, 4, 3)_7 = (2, 6, 2)_8.$$

Irregular consecutive palindromes are rare compared to regular consecutive palindromes. Therefore, in the following chapters, our main focus will be on regular consecutive palindromes.

We suspect that a quadruple palindrome does not exist. This would imply that $k \geq 4$ palindromes do not exist, since every $k+1$ palindrome is necessarily a k palindrome by Definition 1.8.

We have contributed with some results on consecutive double and triple palindromes, to the OEIS (`https://oeis.org`), under the sequences `A279092` and `A279093`, respectively, under the COMMENTS. These results will be presented in following chapters.

Searching for simultaneous and consecutive palindromes is equivalent to searching for normal forms that are palindromic.

If n is not fixed, it is not trivial to find a normal form of a number n in some base b_2 for a given its normal form in base $b_1 \neq b_2$. To find normal forms, we introduce an intermediate step with the following two definitions. We will apply this in the linearization section.

Definition 1.9. *Let $n, b \in \mathbb{N}$ and $b \geq 2$. If there exist integers $d \in \mathbb{N}, \alpha_i \in \mathbb{Z}, i \in [1, d]$ such that*

$$n = \sum_{i=1}^{d} \alpha_i b^{d-i} = (\alpha_1, \alpha_2, \ldots, \alpha_d)_b,$$

then the rightmost expression is called a general form of the number n in the number base b.

Specifically, if the general form is distinct from the corresponding base b normal form, then it will be referred to as a irregular form of the number n in base b. That is, if there exists i such that $\alpha_i \neq a_i$ where $a_i, i \in [1, d]$ are digits of the base b normal form of n, then we call $(\alpha_1, \alpha_2, \ldots, \alpha_d)_b$ an irregular form of the number n in base b.

Notice that the definition of the general form is similar to the normal form from Definition 1.1, but we dropped the condition $a_i \in (0, b], a_1 \neq 0$. The general form is not unique. For given n, b, there are infinitely many general (irregular) forms. Due to Proposition 1.3, all general forms except one, are irregular forms.

Observe that in the definition of the general form, for some $o_i \in \mathbb{Z}, i \in [1, d]$, we have $\alpha_i b^i = (\alpha_i \pm o_i)b^i \mp (o_i b)b^{i-1}$. We can "carry over" multiples of b from consecutive digits by changing the digits by $\pm o_i$ parameters, without changing the value of n.

This gives us a bijection from some tuples $(o_0, o_1, \ldots, o_{d-1}) \in \mathbb{Z}^d$ and some general form, to all other general forms with at most $d+1$ digits. This motivates the following definition.

Definition 1.10. *Let $(a_1, \ldots, a_d)_b$ be some d digit general form of the number n in base b. We define $(o_0, o_1, \ldots, o_{d-1}) \in \mathbb{Z}^d$ as carry (over) parameters, such that*

$$\begin{aligned} n &= (a_1, a_2, a, a_{d-1}, a_d)_b \\ &= (0, a_1, a_2, \ldots, a_{d-1}, a_d)_b \\ &= (o_0, a_1 + o_1 - o_0 b, a_2 + o_2 - o_1 b, \ldots, a_{d-1} + o_{d+1} - o_{d-2} b, a_d - o_{d-1} b)_b. \end{aligned}$$

The last line in the above equality sequence is referred to as the carried form, by the corresponding carry (over) parameters $(o_0, o_1, \ldots, o_{d-1})$.

If the carried form is normal, then it is also called the normalized form. In that case, the carry parameters are referred to as the normalization parameters.

If more than $d+1$ digits are needed, one can define o_i's for $i < 0$ to extend the bijection to general forms of at most $d + 1 + d'$ digits, by introducing d' more leading 0 digits.

Corollary 1.11. *If n has at most $d_0 + 1, d_0 \in \mathbb{N}$ digits in its normal form in some base b, then for any d_0 digit general form of n in the base b, there exists a unique tuple of carry parameters $(o_0, o_1, \ldots, o_{d_0-1})$ such that the carried form is a normal form. That is, normalization parameters of a general form of some n in some base b, are unique.*

The above corollary is true since the normal form is unique as given in Proposition 1.3, and since the carry parameters are in bijection with all such general forms, among which exactly one general form is that unique normal form.

Notice that, individual carry parameters o_i represent the changes applicable to digits of a general form of some number n in base b, such that the value of n is unchanged.

We have translated the problem of finding normal forms to the problem of normalizing arbitrary general forms. This will be applied in Subsection 2.1.1.

2 Regular consecutive palindromes

In the previous chapter, we have defined regular consecutive palindromes as positive integers n that are simultaneously and nontrivially palindromic in $k \geq 2$ number bases $b, b+1, \ldots, b+k-1$, and have exactly d digits in all of those number bases. This means that $2 \leq b \leq n-2$ and $d \geq 2$. It is not hard to see that if n is a consecutive palindrome starting at base b, then $d \geq 2$ implies $b \leq n-2$.

Note that a number n is palindromic in k consecutive number bases if and only if n is a double palindrome in all consecutive pairs of bases $(b, b+1), (b+1, b+2), \ldots, (b+k-2, b+k-1)$. If we find a set of all double palindromes, we could find all $k > 2$ palindromes by intersecting those found solutions. Hence, for now, let us restrict to $k = 2$.

Let $a_1, \ldots, a_d$ and $c_1, \ldots, c_d$ be digits of normal forms of the number n in bases b and $b+1$, respectively. The number n is a double consecutive palindrome if and only if these digits satisfy the following system:

$$\begin{gathered} n = \sum_{i=1}^{d} a_i b^{d-i} = \sum_{i=1}^{d} c_i (b+1)^{d-i} \\ a_i \in [0, b), c_i \in [0, b+1), a_1 \neq 0, c_1 \neq 0, b \geq 2, d \geq 2, i \in [1, d] \\ a_i = a_{d-i+1}, c_i = c_{d-i+1}, \forall i \in [1, d]. \end{gathered} \tag{2.1}$$

Due to Proposition 1.3, we can express the base $b+1$ digits over the base b digits, as they uniquely determine each other. Additionally, digits are palindromic. Therefore, for a fixed $d \geq 2$, we need to determine $\lceil d/2 \rceil$ digits in either number base.

Before linearizing equation (2.1) to obtain the corresponding linear Diophantine systems which we need to solve, first we will prove that solutions do not exist if d is even.

Lemma 2.2. *If $n \in \mathbb{N}$ is a palindrome with an even amount of digits $d = 2l, l \in \mathbb{N}$ in the number base b, then it is divisible by $b+1$.*

Proof:
The proof relies on modular arithmetic. Let $x, y \in \mathbb{N}_0$. Notice that $b^x \equiv (-1)^x \pmod{b+1}$. Then, for x, y of distinct parity, $b^x + b^y \equiv 0 \pmod{b+1}$. Now, observe that if n is an even length palindrome with $d = 2l, l \in \mathbb{N}$ digits, then

$$n = \sum_{i=1}^{2l} a_i b^{2l-1} = \sum_{i=1}^{l} a_i (b^{2l-i} + b^{i-1}) \equiv 0 \pmod{b+1}.$$

□

If n is divisible by $b+1$, then it ends in $c_d = 0$ in base $b+1$, and thus cannot be palindromic, since $c_1 \neq 0$ contradicts with $c_1 = c_d$. Hence, the above lemma implies the following theorem.

Theorem 2.3. *If the system (2.1) has solutions, then the digit case under which it was solved is odd, i.e.,* $d = 2l + 1, l \in \mathbb{N}$.

The implication is, that there are no consecutive palindromes with an even number of digits. So from now on, assume $d = 2l + 1, l \in \mathbb{N}$, which means that the number of digits is odd.

2.1 Linearizing corresponding Diophantine system

The goal is to reduce the system (2.1) to a set of linear Diophantine systems. Such and similar systems and equations are well studied (see [5, 10, 21]). We fix the variable d and apply a linearization method. Afterwards, by solving finitely many systems of linear Diophantine equations, we obtain all regular d digit k-consecutive palindromes. We can solve these systems in software like `Wolfram Mathematica` (see [20]) and `SageMath` (see [2]). Note that generally systems of linear Diophantine equations in $\{+, -, \times, /, \bmod, <\}$ are not necessarily always solvable in strongly polynomial time (see [9]).

Recall that we fixed $k = 2$ for the system (2.1). For $k > 2$, we could set up a similar system and linearize it analogously. But, there is no need for this if we solve $k = 2$, since those solutions are reducible to $k > 2$ solutions, for a fixed d digit case. The case of $k = 3$ solutions obtained by such reduction is presented in Section 2.3.

2.1.1 Carry over parameters linearization method

We will look at the system (2.1), but backward. Instead of bases $b, b + 1$ we consider bases $b, b - 1$ and $b \geq 3$. The only reason for going backward is, to end up with nonnegative digits in the general form we will obtain. We have:

$$\begin{gathered} n = \sum_{i=1}^{d} a_i b^{d-i} = \sum_{i=1}^{d} c_i (b-1)^{d-i} \\ a_i \in [0, b), c_i \in [0, b-1), a_1 \neq 0, c_1 \neq 0, b \geq 2, d \geq 2, i \in [1, d] \\ a_i = a_{d-i+1}, c_i = c_{d-i+1}, \forall i \in [1, d]. \end{gathered} \tag{2.4}$$

The idea is to express digits c_i in terms of digits a_i. Observe that

$$\sum_{i=1}^{d} a_i b^{d-i} = \sum_{i=1}^{d} a_i ((b-1)+1)^{d-i} = \sum_{i=1}^{d} c_i (b-1)^{d-i}.$$

By using the binomial expansion and applying Definition 1.10 in the case $o_0 = 0$, we obtain

$$c_i = \sum_{j=1}^{i} \binom{d-j}{d-i} a_j = \sum_{j=1}^{i} \binom{d-j}{d-i} a_j + o_i - o_{i-1}(b-1). \tag{2.5}$$

Corollary 1.11 implies that there are finitely many $(o_1, \ldots, o_{d-1})$ such that the general form given by $c_i, i \in [1, d]$ is normal.

We can now iterate all of the finitely many values of $(o_1, \dots, o_{d-1})$. This gives us a linear system of Diophantine equations

$$c_i = c_{d-i+1}, \quad i \in [1, d], \tag{2.6}$$

which we have to solve in each iteration, under conditions $a_i \in [0, b), c_i \in [0, b-1), a_1 \neq 0, c_1 \neq 0, b \geq 2$, where $d \geq 2$ is fixed and $a_i = a_{d-i+1}, i \in [1, d]$ (from n). We are solving it for $a_i, i \in [1, d]$ and b. The upper and lower bounds on individual o_i are not immediately clear.

For example, one family of solutions to these systems is explicitly known and given in equation (2.11). The special property of this example is that it is not limited by fixed cases of d, but instead spans across arbitrary values of digits d.

2.1.2 Free polynomial term linearization method

This method was suggested to us by Alekseyev on Math Overflow (see [2]), where he gives an example in the case of $d = 5$ and bases $b, b-1$. Here we will observe d in general and bases $b, b+1$. Compared to the previous method, the obtained systems will be equivalent. But, the upside here is, that we get bounds on introduced parameters immediately.

The idea here is to look at the equality in the system (2.1) as equality of polynomials $P(b) = P(b+1)$ and to observe $P(b+1) - P(b) = 0$. We write the left-hand side as a unified polynomial $P(b+1) - P(b) = T(b)$ in terms of b. The binomial expansion gives us $T(b)$ explicitly, and we are left with the equality:

$$T(b) = \sum_{i=1}^{d} t_i b^{d-i} = \sum_{i=1}^{d} \left(\sum_{j=1}^{i} \binom{d-j}{d-i} c_j - a_i \right) b^{d-i} = 0. \tag{2.7}$$

Let $d = 2l + 1, l \in \mathbb{N}$. For $i \in [1, d]$ we have the palindromic conditions

$$a_i = a_{d-i+1},$$

$$c_i = c_{d-i+1},$$

and normal form conditions ($b \geq 2$ is a number base)

$$a_i \in [0, b) = [0, b-1], a_1 \neq 0,$$

$$c_i \in [0, b+1) = [0, b], c_1 \neq 0.$$

Now we will linearize this by obtaining bounds on coefficients $t_d, t_{d-1}, \dots, t_1$ of $T(b)$. We will introduce parameters $k_1, k_2, \dots, k_{d-1}$ whose bounds will follow from the coefficient bounds. Those parameters will be used to replace t_{d-i+1} with $k_i b$, for $i \in [1, d]$, so we can apply division by b on equality (2.7).

We start by extracting bounds of t_d, the free term of $T(b)$. Applying the palindromic conditions on (2.7), we obtain

$$t_d = 2 \sum_{j=1}^{l} c_j + c_{l+1} - a_1.$$

Now, applying the normal form conditions, we get the bounds on t_d, i.e.,

$$t_d \in 2[1,b] + 2\sum_{j=2}^{l}[0,b] + [0,b] - [b-1,1] = [3-b,(2l+1)b-1] .$$

Since $T(b) = 0$, the free term t_d is divisible by b. This implies that $t_d = k_1 b$, for some integer $k_1 \in \mathbb{Z}$. From the conditions on t_d we have

$$\frac{3}{b} - 1 \leq k_1 \leq 2l + 1 - \frac{1}{b} \implies k_1 \in [0, 2l] =: K_1.$$

Now, we replace t_d by $k_1 b$ in equation (2.7), and divide it by b. The new free term will be $k_1 + t_{d-1}$.

We repeat this process with the new free term. At the end, we obtain again a new parameter $k_2 \in K_2$ and a new free term $k_2 + t_{d-2}$. This procedure is continued until the $(d-1)$th step, where we end up with $k_{d-1} + t_1 = 0$. At the end, this process gives us finitely many linear Diophantine systems

$$\begin{cases} t_d = k_1 b \\ k_1 + t_{d-1} = k_2 b \\ k_2 + t_{d-2} = k_3 b \\ \vdots \\ k_{d-2} + t_2 = k_{d-1} b \\ k_{d-1} + t_1 = 0 \end{cases}, \tag{2.8}$$

by iterating over all possible combinations of $k_1, \ldots, k_{d-1}$ parameters, whose upper and lower bounds will be known $k_i \in K_i \subset \mathbb{Z}, i \in [1, d-1]$. “Together with the bounding conditions for a_i and c_i, each such system defines a polyhedron (possibly unbounded), whose integer points can be found with existing algorithms” - Alekseyev (see [2]).

We have obtained a set of linear Diophantine systems in variables $a_i, c_i, i \in [1, l+1]$ and b, determined by $d-1$ additional parameters $k_i \in K_i, i \in [1, d-1]$, where $d = 2l+1, l \in \mathbb{N}$ is fixed. In comparison to the first method, these parameters would be analogous to $o_i, i \in [1, d-1]$ carry parameters.

Notice that this method is nothing more, but an extension to the method from Subsection 2.1.1, but now, we additionally and immediately know bounds on introduced parameters. However, we keep in mind that these obtained parameters are not optimal. That is, only a fraction of their combinations will give systems that have solutions. To solve greater digit cases d in a reasonable time, these parameters need to be optimized.

Summarizing the methods known so far, we have the following corollary.

Corollary 2.9. *Solving systems (2.1) and (2.4) for some fixed $d = 2l+1, l \in \mathbb{N}$, is equivalent to solving all linear Diophantine systems given by either systems* (2.5) *and* (2.6) *or by systems* (2.7) *and* (2.8).

This corollary tells us how we will obtain regular double palindromes in the next section. Keep in mind that the Subsection 2.1.1 was stated in terms of bases $b, b-1$ and that the Subsection 2.1.2 was stated in terms of bases $b, b+1$. Hence, digits represented by a_i, c_i are swapped, and the bases are offset by one.

Solving these systems we obtain all solutions for some fixed values d. It is not easy to find the general solution for all digit cases d (a "closed form" of all such palindromes). However, there could be means to obtain infinite families of solutions spanning across arbitrary digit cases d. At the beginning of the next section, first we will present one such family. It was found by speculation and then proven by construction. We are not sure how to obtain more such families of solutions covering non-fixed values d.

2.2 Two number bases

In this section, we will consider solving of systems of linear Diophantine equations obtained from applying the linearization methods presented in the previous section, to obtain regular double palindromes ($k = 2$). Theorem 2.3 implies that we only need to consider odd digit cases, i.e., $d = 2l + 1, l \in \mathbb{N}$.

Before we start, we will present a result: A family of solutions to systems given in Corollary 2.9 is known, which yields infinitely many solutions for every (odd) digit case. We would also strongly conjecture, that when ignoring this family of solutions, we are still left with infinitely many solutions for every (odd) digit case.

We acknowledge that the identity in the following theorem was suggested by the user named `Peter`. We subsequently presented the proof, under the username `Vepir`. Both posts are available on the Mathematics Stack Exchange (MSE) website, under the question "Arbitrarily long palindromes in two consecutive number bases" (see [19]).

Theorem 2.10 (The trivial family of regular double palindromes).
Let $l \in \mathbb{N}, l > 1$. The number

$$n = \frac{b^{2l} - 1}{b + 1}$$

is a $d = 2l - 1$ digit double palindrome in bases $b, b+1$, for all $b \geq \sum_{s=1}^{l} \binom{l}{s}^2 = \binom{2l}{l} - 1$.

Proof:
The proof is constructive. Firstly, it is not hard to see that the given n is palindromic in base b:

$$\frac{b^{2l} - 1}{b + 1} = (b-1, 0, b-1, 0, \dots, 0, b-1, 0, b-1)_b.$$

To prove that it is also palindromic in base $b+1$, we define the symmetric (palindromic) digits

$$A_l(i) = \begin{cases} b - \binom{2l}{2l-i} + 1, & i \text{ is odd} \\ \binom{2l}{2l-i} - 1, & i \text{ is even} \end{cases},$$

where $i \in [1, 2l-1]$. Notice that $A_l(i)$ is maximal at $i = l$, which gives $A_l(i) \leq \binom{2l}{l} - 1 < b+1$. It is clear that $A_l(i) \geq 0$, for $i \in [1, 2l-1]$. This implies that the palindromic form

$$(A_l(1), A_l(2), \ldots, A_l(2l-1))_{b+1}$$

is a valid $2l-1$ normal form of some number m in base $b+1$, for all $b \geq \binom{2l}{l} - 1$. What is left to show, is that $m = n$. That is, we need to prove that the normal form sum of m (see Definition 1.1) is given by the closed form expression of n, i.e.,

$$m = (A_l(1), A_l(2), \ldots, A_l(2l-1))_{b+1} = \sum_{i=1}^{2l-1} A_l(i)(b+1)^{2l-1-i} = \frac{b^{2l}-1}{b+1} = n.$$

This is easily verified with a Computer Algebra System (CAS) like `Wolfram Mathematica`.

□

The `Wolfram Mathematica` code that will verify the end of the above proof, can be run in the freely accessible Wolfram Lab if the license of the mentioned software is not available to the reader. It is given with

```
A[i_, l _] := (b ((-1)^(i+1)+1)/2 + (-1)^i (Binomial[2l,-i+2l]-1));
s = Sum[A[i, l ] (b + 1)^(2 l - 1 - i), {i, 1, 2 l - 1}];
FullSimplify[s, Element[l, Integers]].
```

We can rewrite results from Theorem 2.10 in the context of systems obtained by linearization methods from Section 2.1.

That is, we use the first linearization method presented in Subsection 2.1.1. Here we have bases $b, b-1$ instead of bases $b, b+1$ as originally stated in the theorem, since the system (2.6) was obtained by linearizing the system (2.4). We have family from Theorem 2.10 in this context:

$$\left(a_i = \begin{cases} b - \binom{2l_0}{2l_0 - i}, & i \text{ is odd} \\ \binom{2l_0}{2l_0 - i} - 1, & i \text{ is even} \end{cases} : i \in [1, l_0], l_0 > 1; b \geq \binom{2l_0}{l_0} \right)_b, \tag{2.11}$$

over all such l_0, b and all odd d. Now, parameters $o_i, i \in [1, d-1]$ can be backtracked and deduced, for this particular set of solutions, for every case of $l_0 = 2, 3, 4, 5, \ldots$:

$$(o_1, \ldots, o_{d-1}) = (2,1), (4,6,6,2), (6,15,24,21,12,3), (8,28,62,85,80,49,20,4), \ldots,$$

corresponding to $l = 1, 2, 3, 4, \ldots$, that is, to the $l = l_0 - 1, d = 2l+1$ digit regular double palindromes. Similarly, we could rewrite the family in the context of the other linearization method from Subsection 2.1.2.

Theorem 2.10 directly implies the following corollary:

Corollary 2.12. *For every fixed $d = 2l+1, l \in \mathbb{N}$ odd case of digits, there are infinitely many d digit regular double palindromes.*

This implies that consecutive palindromes can be arbitrarily long, in terms of digits. As mentioned earlier, we strongly conjecture that there are infinitely many regular double palindromes, for every fixed odd case of digits, other than those provided by the identity in Theorem 2.10. Finding a closed form for all of them looks hard.

Problem 2.13 (Open Problem). *Can we find more families of regular double palindromes that span across arbitrary digit cases d, other than the one given in Theorem 2.10?*

Perhaps someone interested in this topic can establish more such families.

We are not able to construct more such families nor solve the problem of finding all regular double palindromes generally for all digits. Hence, we decided to fix the digits variable d. This allows us to apply the proposed linearizations from Subsections 2.1.1, 2.1.2 to obtain double palindromes for fixed digit cases d.

2.2.1 Three digits case

In this subsection, we will determine all $d = 3$ ($l = 1$) regular double palindromes. We have previously shared this result under the OEIS sequence `A279092` and on Math Overflow (see [18]). When we solve either set of linear Diophatine systems given in Corollary 2.9 for $d = 3$, we obtain the following:

Theorem 2.14 (Three digit regular double palindromes). *The number $n \in \mathbb{N}$ is a regular $d \le 3$ digit double palindrome if and only if it belongs to one of the following two normal form families:*

$$\begin{aligned} n_1 &= (x+4, 0, x+4)_{x+5} &&= (x+2, 5, x+2)_{x+6} \\ n_2 &= (x+2, x+3, x+2)_{x+y+4} &&= (x+1, y+4, x+1)_{x+y+5} \end{aligned},$$

were $x, y \in \mathbb{N}_0$.

The proof of Theorem 2.3 is structured as follows:

Firstly, $d \ge 2$ holds by Definition 1.8. Secondly, $d = 2$ does not have solutions by Theorem 2.3. Finally, by solving systems from Corollary 2.9 for $d = 3$ we obtain normal forms from Theorem 2.14. We will demonstrate how to solve systems from Corollary 2.9 for $d = 3$ using both linearization methods. Looking at either bases $b, b+1$ or bases $b, b-1$, we need to find all $n \in \{n_1, n_2, \dots\}$ such that

$$n = (a_1, a_2, a_1)_b = (c_1, c_2, c_1)_{b \pm 1}.$$

First, we show how to solve this with the second linearization approach, entitled with *Proof I.* Here we simply use a computer to solve the obtained system.

Afterward, we again solve this but now with the first linearization approach, entitled with *Proof II.* Here we use the idea of normalization presented in Definition 1.10.

Proof I.
Here we use the polynomial method given in Subsection 2.1.2 to obtain the systems of linear Diophantine equations. The introduced parameters k_1, k_2 will immediately be given bounds. We have the following system:

$$\begin{cases} 0 + 2c_1 + c_2 - a_1 = k_1 b, k_1 \in [0,2] \\ k_1 + 2c_1 + c_2 - a_2 = k_2 b, k_2 \in [0,3] \\ k_2 + c_1 + a_1 = 0 \end{cases}, \tag{2.15}$$

where $a_1 \in [1, b-1], a_2 \in [0, b-1], c_1 \in [1, b], c_2 \in [0, b], b \geq 2$.

Now, we can solve the $3 \cdot 4 = 12$ corresponding systems of linear Diophantine equations (iterate and fix k_1, k_2 to obtain individual systems) with existing algorithms (Section 2.1). Alternatively, in this particular case of a small value d, these systems can be directly solved with `Wolfram Mathematica`.

We get that the only solutions are n_1, n_2 as given in Theorem 2.14, and we are done. We also get that pairs (k_1, k_2) corresponding to those solutions are $(1,2), (1,1)$, respectively. □

Out of 12 systems produced by approach from Subsection 2.1.2, only 2 of them had solutions. For cases of greater d, there is a significant increase in the number of solutionless systems. This leads us to consider if further optimization of approach from Subsection 2.1.2 is possible.

In the second proof, we have a slightly longer story. Unlike the bounded k_i parameters in Subsection 2.1.2, the bounds on parameters o_i are not given in Subsection 2.1.1.

Proof II.
Here we use the carry over method given in Subsection 2.1.1. It gives us one base $b-1$ general form

$$n = (a_1, a_2, a_1)_b = (c_1, c_2, c_3)_{b-1} = (a_1, 2a_1 + a_2, 2a_1 + a_2)_{b-1},$$

where we need to satisfy $a_1 \in [1, b-1], a_2 \in [0, b-1], c_1 \in [1, b-2], c_2 \in [0, b-2], b \geq 2$ and $c_1 = c_3$. We can express conditions on c_1, c_2, c_3 over a_1, a_2:

$$\begin{cases} c_1 = a_1 + o_1 \in [1, b-2] \\ c_2 = 2a_1 + a_2 - o_1(b-1) + o_2 \in [1, b-2] \\ c_3 = 2a_1 + a_2 - o_2(b-1) \in [0, b-2] \end{cases},$$

and observe that

$$c_1 = c_3 \iff a_1 + o_1 = 2a_1 + a_2 - o_2(b-1).$$

First we want to establish all carry parameters $O_2 := (o_1, o_2)$ under which the system could have solutions. That is, set up our parameter bounds.

Firstly, notice that $O_2 = (0,0)$ does not have solutions since $c_1 = a_1 < 2a_1 + a_2 = c_3$, i.e., the unit digit c_3 is too large. Additionally, observe that decreasing either or both o_1, o_2 will make the difference $|c_1 - c_3|$ greater. This implies that o_1, o_2 must be nonnegative.

Since $c_2 = c_3$, when $o_1 = o_2 = 0$, we have that $O_2 \notin \{(1,0),(0,1)\}$. Otherwise, one of the c_2, c_3 will be at least $b-1$ if the other one is not. Now, we have the lower bounds on carry over parameters $o_1, o_2 \geq 1$.

Notice that base $b \geq \max\{a_1, a_2\} + 1$, since $a_1, a_2 < b$. Substituting this, we get that if $o_2 \geq 3$, then $c_3 \leq 0$, which contradicts with $c_3 = c_1 \neq 0$, implying $o_2 \leq 2$.

Recalling $b-1 \geq 2, o_2 \leq 2$ and assuming $o_1 \geq 4$ we obtain $c_2 \leq -(b-1)+2$, implying $o_1 \leq 3$. Summing this up, we are left with $O_2 \in \{(1,1),(1,2),(2,1),(2,2),(3,1),(3,2)\}$. Finally, we can eliminate the case $O_2 = (3,1)$, since $c_3 < (b-1)$ implies $2a_1 + a_2 < 2(b-1)$, implying $c_2 < -(b-1)+1$ which is negative.

After obtaining $O_2 \in \{(1,1),(1,2),(2,1),(2,2),(3,2)\}$, we can analyze each case individually and apply the analysis of the normal forms we defined. This avoids solving systems of linear Diophantine equations altogether, for this particularly small case of value d.

- In the case of $O_2 = (2,1)$, notice that the solution of the system is given in Theorem 2.10, when looking at it in the context of carry parameters (2.11) when $l_0 = 2$ ($l = 1, d = 3$). This solution corresponds to n_1 in Theorem 2.14.

What is left, is to find n_2 and show that other cases do not have additional solutions. Define the base $b_0 = b - 1 = \max\{a_1, a_2\} + \beta_0, \beta_0 \in \mathbb{N}$ to satisfy conditions $a_1, a_2 < b$.

- In the case of $O_2 = (1,1)$, we have

  $$n = (a_1 + 1, 2a_1 + a_2 - b_0 + 1, 2a_1 + a_2 - b_0)_{b_0},$$

  where $a_1 + 1 = 2a_1 + a_2 - b_0 \iff a_2 = b_0 - a_1 + 1$. This is not a problem, since $a_2 < b \iff b - 1 - a_1 + 1 < b \iff a_1 > 0$. This gives

  $$n = (a_1 + 1, a_1 + 2, a_1 + 1)_{b_0}.$$

  Notice that this base b_0 form does not depend on a_2 anymore. We only need $a_1 + 2 < b_0$, so we write $b_0 = a_1 + 2 + \beta, \beta \in \mathbb{N}$. We replace a_1 with $x \in \mathbb{N}_0$ and $2 + \beta$ with $y \in \mathbb{N}_0$ to finally obtain the solution to this case, i.e.,

  $$n = (x + 2, x + 3, x + 2)_{x+y+4},$$

  which gives n_2 in Theorem 2.14.

- In cases $O_2 = (1,2),(2,2),(3,2)$, we have $o_2 = 2$. This will lead to

  $$a_1 + o_1 = 2a_1 + a_2 - 2b_0 \iff a_2 = 2b_0 - a_1 + o_1.$$

  But this is a contradiction. Recall that $b_0 = b - 1$ and that $a_2 < b$, so we have

  $$2(b-1) - a_1 + o_1 < b \iff a_1 \geq b + (o_1 - 1),$$

  which contradicts with $a_1 < b$, for $o_1 \geq 1$.

We've obtained n_1, n_2 and shown that $n_3, n_4, n_5, \ldots$ do not exist. This finishes the proof. $\square$

2.2.2 Five digits case

In this subsection, we will present all $d = 5$ ($l = 2$) regular double palindromes. We have previously shared this result on Math Overflow, but in bases $b, b-1$ (see [18]).

Theorem 2.16 (Five digit regular double palindromes)**.** *The number $n = (a_1, a_2, a_3, a_2, a_1)_b$ is a regular $d = 5$ digit double palindrome in number bases $b, b+1$ if and only if base b and its digits belong to one of the following cases:*

(k_1, k_2, k_3, k_4)	a_1	a_2	a_3	b
$(1,2,3,1)^*$	$4+x+y$	$1+2a_1$	$-b+3a_1$	$10+3x+2y$
$(1,3,3,2)^*$	$14+x$	$1-b+2a_1$	$1-b+3a_1$	$2a_1-6$
	$12+x$	$1-b+2a_1$	$1-b+3a_1$	$2a_1-5$
	$10+x$	$1-b+2a_1$	$1-b+3a_1$	$2a_1-4$
	$8+x$	$1-b+2a_1$	$1-b+3a_1$	$2a_1-3$
	$6+x$	$1-b+2a_1$	$1-b+3a_1$	$2a_1-2$
	$7+x$	$1-b+2a_1$	$1-b+3a_1$	$2a_1-1$
	$8+x$	$1-b+2a_1$	$1-b+3a_1$	$2a_1\pm 0$
	$9+x$	$1-b+2a_1$	$1-b+3a_1$	$2a_1+1$
$(1,3,4,2)$	$[4,6]$	$1-b+2a_1$	$-2b+3a_1$	a_1+1
	$[4,7]$	$1-b+2a_1$	$-2b+3a_1$	a_1+2
	$[6,8]$	$1-b+2a_1$	$-2b+3a_1$	a_1+3
	$[8,9]$	$1-b+2a_1$	$-2b+3a_1$	a_1+4
	10	$1-b+2a_1$	$-2b+3a_1$	a_1+5
$(2,4,3,1)^*$	$2+x$	$2+2a_1$	$3+3a_1$	$10+3x+y$
$(2,4,4,1)^*$	$13+3x$	$2+2a_1$	$2-b+3a_1$	$3a_1-2$
	$11+3x$	$2+2a_1$	$2-b+3a_1$	$3a_1-1$
	$9+3x$	$2+2a_1$	$2-b+3a_1$	$3a_1\pm 0$
	$7+3x$	$2+2a_1$	$2-b+3a_1$	$3a_1+1$
	$8+3x$	$2+2a_1$	$2-b+3a_1$	$3a_1+2$
$(2,5,4,2)$	$[6,7]$	$2-b+2a_1$	$3-b+3a_1$	$2a_1-1$
	$[4,8]$	$2-b+2a_1$	$3-b+3a_1$	$2a_1\pm 0$
	$[3,9]$	$2-b+2a_1$	$3-b+3a_1$	$2a_1+1$
	$[3,10]$	$2-b+2a_1$	$3-b+3a_1$	$2a_1+2$
$(2,5,5,2)$	4	$2-b+2a_1$	$2-2b+3a_1$	$2a_1-2$
	$[3,4]$	$2-b+2a_1$	$2-2b+3a_1$	$2a_1-1$
$(2,5,6,3)^*$	$14+2x+y$	$2-b+2a_1$	$1-2b+3a_1$	$16+3x+y$
$(2,6,6,4)^*$	$18+x$	$2-2b+2a_1$	$2-2b+3a_1$	$19+x$
$(3,7,6,3)^*$	$23+2x+y$	$3-b+2a_1$	$4-b+3a_1$	$37+3x+2y$
$(3,7,7,3)^*$	$[10,19]+2x$	$3-b+2a_1$	$3-2b+3a_1$	$2a_1-x-8$
	$[11,14]+2x$	$3-b+2a_1$	$3-2b+3a_1$	$2a_1+x+2$
$(3,8,8,4)$	$[12,16]$	$3-2b+2a_1$	$3-3b+3a_1$	a_1+1

.

Table 1: Base b digits of all bases $b, b+1$ five digit regular double palindromes.

Here $x, y \in \mathbb{N}_0$ and $[z_1, z_2]$ goes over finitely many integers. Cases $(k_1, k_2, k_3, k_4)^*$ generate infinitely many solutions and are called infinite families of solutions. The remaining cases (k_1, k_2, k_3, k_4) generate finitely many solutions and are called finite families of solutions.

The following table gives the corresponding base $b+1$ digits $(c_1, c_2, c_3, c_2, c_1)_{b+1}$, in terms of b and a_1 from the above Table 1

(k_1, k_2, k_3, k_4)	c_1	c_2	c_3
$(1, 2, 3, 1)^*$	$-1 + a_1$	$2 + b - 2a_1$	$-2 - b + 3a_1$
$(1, 3, 3, 2)^*$	$-2 + a_1$	$6 + b - 2a_1$	$-8 - b + 3a_1$
$(1, 3, 4, 2)$	$-2 + a_1$	$5 + b - 2a_1$	$-6 - b + 3a_1$
$(2, 4, 3, 1)^*$	$-1 + a_1$	$3 + b - 2a_1$	$-4 + 3a_1$
$(2, 4, 4, 1)^*$	$-1 + a_1$	$2 + b - 2a_1$	$-2 + 3a_1$
$(2, 5, 4, 2)$	$-2 + a_1$	$6 + b - 2a_1$	$-8 + 3a_1$
$(2, 5, 5, 2)$	$-2 + a_1$	$5 + b - 2a_1$	$-6 + 3a_1$
$(2, 5, 6, 3)^*$	$-3 + a_1$	$8 + 2b - 2a_1$	$-10 - 2b + 3a_1$
$(2, 6, 6, 4)^*$	$-4 + a_1$	$12 + 2b - 2a_1$	$-16 - 2b + 3a_1$
$(3, 7, 6, 3)^*$	$-3 + a_1$	$9 + 2b - 2a_1$	$-12 - b + 3a_1$
$(3, 7, 7, 3)^*$	$-3 + a_1$	$8 + 2b - 2a_1$	$-10 - b + 3a_1$
$(3, 8, 8, 4)$	$-4 + a_1$	$11 + 2b - 2a_1$	$-14 - b + 3a_1$

Table 2: Base $b + 1$ digits of all bases $b, b + 1$ five digit regular double palindromes.

Notice that there are 12 cases of (k_1, k_2, k_3, k_4) parameters under which solutions exist, compared to only 2 such cases in the three digit Theorem 2.14. Observe that the $d = 5$ solutions given by the trivial family Theorem 2.10 represent the $(k_1, k_2, k_3, k_4) = (2, 6, 6, 4)$ case. This is seen in equation (2.11) in terms of $(o_1, o_2, o_3, o_4) = (4, 6, 6, 2)$ parameters, when we substitute in $d = 5$ ($l_0 = 3 \iff l = 2$).

Theorem 2.16 was obtained by solving systems from Subsection 2.1.2 for $d = 5$. That is, finding regular five-digit double palindromes is equivalent to solving the following set of linear Diophantine systems generated by $k_i, i \in [1, 4]$ parameters:

$$\begin{cases} 0 - a_1 + 2c_1 + 2c_2 + c_3 = k_1 b, k_1 \in [0, 4] \\ k_1 - a_2 + 4c_1 + 4c_2 + 2c_3 = k_2 b, k_2 \in [0, 10] \\ k_2 - a_3 + 6c_1 + 3c_2 + c_3 = k_3 b, k_3 \in [0, 10] \\ k_3 - a_2 + 4c_1 + c_2 = k_4 b, k_4 \in [0, 5] \\ k_4 - a_1 + c_1 = 0 \end{cases}, \tag{2.17}$$

where $a_1 \in [1, b-1], a_2, a_3 \in [0, b-1], c_1 \in [1, b], c_2, c_3 \in [0, b], b \geq 2$.

This can be solved by applying existing algorithms (Section 2.1), for example, in `SageMath` (see [2]). Additionally, since this case of digits $d = 5$ is relatively small, it is also directly solvable with `Wolfram Mathematica`'s general purpose `Reduce[]` method, in reasonable time. For cases $d \geq 7$, the computational time of `Reduce[]` is too long.

The computational time can be optimized by optimizing bounds on $k_i, i \in [1, d-1]$ parameters before applying the solving procedure on them. Notice that only a fraction of cases given by iterating all $k_i, i \in [1, d-1]$ actually has solutions. In the current case of $d = 5$ digits, we are looking to solve $5 \cdot 11 \cdot 11 \cdot 6 = 3630$ systems of linear Diophantine equations, out of which only 12 have solutions.

We will provide the direct (not optimized) code for `Wolfram Mathematica 11.2` that directly uses the general-purpose `Reduce[]` method. This can be used to verify Theorem 2.16 in a matter of minutes. The following code solves all systems given by (2.17):

```
ClearAll[a, c, b, l, x, eq, bUp, bLow, sys];
x[a_, b_, l_] := Sum[a[i+1] (b^i + b^(2l-i)), {i, 0, l-1}] + a[l+1] b^l;
eq[l_] := Fold[And, MapIndexed[#1 == k[First[#2]] b &, CoefficientList[
     x[c, b + 1, l] - x[a, b, l] + Sum[k[i] b^i, {i, 0, 2 l}], b]]
     /. {k[0] -> 0, k[2 l + 1] -> 0}];
bUp[l_] :=  Fold[And, Join[Table[a[i + 1] < b, {i, 0, l}],
     Table[c[i + 1] < b + 1, {i, 0, l}]]];
bLow[l_] := Fold[And, Table[If[i == 0,
     a[i + 1] >= 1 && c[i + 1] >= 1,
     a[i + 1] >= 0 && c[i + 1] >= 0], {i, 0, l}]];
sys[l_] := eq[l] && bUp[l] && bLow[l];
i=0; Timing[Do[ r = Reduce[sys[2] && b >= 2 && (k[1] == k1) &&
(k[2] == k2) && (k[3] == k3) && (k[4] == k4), Variables[sys[2]], Integers];
  If[Length[r] > 0, i += 1; Print[i, " ", {k1, k2, k3, k4}, " ", r]],
  {k1, 0, 4}, {k2, 0, 10}, {k3, 0, 10}, {k4, 0, 5}]].
```

Solutions in the output will be formatted similarly as stated in Theorem 2.16.

2.2.3 Seven or more digits

Individual $d \geq 7$ ($l \geq 3$) cases can be obtained (computed) by applying existing algorithms (Section 2.1), to linear Diophantine systems given in Subsection 2.1.2. To be more precise, this can be automated in `SageMath` (see [2]). The process becomes computationally harder for every next case of digits d.

We will not be explicitly listing any $d \geq 7$ double palindrome solutions in this paper. Efficiently generating and collecting solutions of such systems beyond this point becomes a problem of its own. For example, considering systems of Subsection 2.1.2 in cases $d = 7, 9, 11$ would require processing (solving) $\approx 10^7, 10^{13}, 10^{20}$ linear systems of Diophantine equations, assuming we do not optimize bounds on $k_i, i \in [1, d-1]$ parameters. Additionally, the solution cases get more numerous and harder to write down in a compact form.

Instead of collecting finitely many d cases, we would like to put an emphasis on solving this problem in a more general sense. That is, finding all regular double palindromes for all d remains an open problem. A weaker variation of this problem is to find isolated families of solutions that would span across arbitrarily large cases of d. This is proposed in Problem 2.13. So far, only one such family is known, and is given in Theorem 2.10. Finding more such families is an open problem.

Summarizing that, we shift our interest from regular double palindromes to regular triple palindromes. Their structure of solutions is much more compact as they are much rarer. They are discussed in the following section.

2.3 Three number bases

In the previous chapter, we discussed regular double palindromes ($k = 2$). We found that individual cases of digits d can be solved by applying the approach from Subsection 2.1.2 such as we demonstrated with $d = 3, 5$. We do not know if it is possible to do the same for a general d. The closest result to a general solution we have is in Theorem 2.10 which gives only the so-called trivial family of solutions.

Hence, in this chapter, we will try to find out if the regular triple palindromes ($k = 3$), could have a nice closed form. This chapter will be brief. By using what was presented in the previous chapter, we computationally solved the problem for $d = 3, 5, 7$ double palindromes. To get $d = 3, 5, 7$ triple palindromes in number bases $b, b+1, b+2$, we simply solve the problem for the intersection of bases $b, b+1$ and bases $b+1, b+2$ double palindromes. For $d \geq 9$, it appears that there are no additional solutions. Recall that the number of digits d must be odd as it is given in Theorem 2.3.

2.3.1 Short digit cases

We say that a regular triple palindrome is "short" if it has $d \leq 7$ digits. To obtain all short regular triple palindromes, we look at the consecutive pairs of bases of solutions of regular double palindromes and find their intersection. We have solved this computationally and shared the obtained result on Mathematics Stack Exchange (see [17]). That is, we have obtained the following result:

Theorem 2.18 (Short regular triple palindromes)**.** *The number $n \in \mathbb{N}$ is a $d \leq 7$ digit regular triple palindrome in bases $b, b-1, b-2$ if and only if it belongs to one of the 9 families of normal forms or is one of the 13 non-family examples. Let $d' = (d+1)/2$. To write down all families, we use the following notation:*

$$(a_1 + \alpha_1 t, \dots, a_{d'} + \alpha_{d'} t, \dots, a_d + \alpha_d t)_b = (a_1, \dots, a_{d'}) + (\alpha_1, \dots, \alpha_{d'})t = (a_i) + (\alpha_i)t.$$

Let $t \in \mathbb{N}_0$. The 9 families (in base b) are given by the following sets of normal forms:

d	(a_i)	$(\alpha_i)t$	b
$d=3$	$(2,6)$	$(1,1)t$	$2t+8$
$d=5$	$(31,32,0)$	$(3,2,1)t$	$4t+47$
$d=7$	$(34,50,10,74)$	$(1,1,1,1)t$	$2t+76$
$d=7$	$(8,33,0,41)$	$(1,3,1,3)t$	$6t+58$
$d=7$	$(112,15,0,36)$	$(4,0,1,0)t$	$6t+175$
$d=7$	$(227,160,187,200)$	$(5,3,5,3)t$	$6t+280$
$d=7$	$(5,23,6,14)$	$(2,6,5,0)t$	$12t+39$
$d=7$	$(93,78,30,50)$	$(10,6,7,0)t$	$12t+119$
$d=7$	$(47,150,249,26)$	$(2,6,11,0)t$	$12t+291$

Table 3: Base b digits of short families of regular triple palindromes in bases $b, b-1, b-2$.

The non-family examples are:

- *If* $d = 7$, *then* $n = 3360633, 19987816, 43443858, 532083314, 1778140759, 2721194733, 11325719295, 47622367425, 97638433343, 224678540182, 265282702996, 561091062285$ *for* $b = 11, 15, 17, 24, 28, 30, 30, 38, 42, 44, 45, 50$, *respectively.*

- *If* $d = 5$, *then non-family examples do not exist.*

- *If* $d = 3$, *then the only non-family example is* $n = 300$ *for* $b = 9$.

For example, the first family "$(2, 6) + (1, 1)t = (2 + t, 6 + t, 2 + t)_b, b = 2t + 8$" for $t = 0$ gives the smallest example $n = (2, 6, 2)_8 = (3, 4, 3)_7 = (4, 5, 4)_6 = 178$ which is a regular triple palindrome in number bases $8, 7, 6$.

Notice that families in Theorem 2.18 are given as compact closed forms. For example, all $d = 5$ regular triple palindromes are given by the family "$(31, 32, 0) + (3, 2, 1)t, b = 2t + 8$". In comparison, all $d = 5$ regular double palindromes do not have a compact closed form and are instead given by the Table 1.

We can rewrite families from Theorem 2.18 in their numerical form $n = f(b)$, i.e., as polynomials in b. We can also write the digits from all three bases $b, b + 1, b + 2$ in the $(a_i) + (\alpha_i)t$ notation. We say that a family given by corresponding $f(b)$ is p-periodic if $b = b_0 + pt$ for some constants b_0, p.

For example, the three digit family is 2-periodic:

$(d = 3)$ $2 - \text{periodic} f(b)$

$b = 8 + 2t$	$f(b) = \frac{1}{2}(b^3 - 3b^2 + 5b - 4)$
$b + 0$	$(4\ \ 5\ \ 4)\ +\ (1\ \ 1\ \ 1)t$
$b + 1$	$(3\ \ 4\ \ 3)\ +\ (1\ \ 1\ \ 1)t$
$b + 2$	$(2\ \ 6\ \ 2)\ +\ (1\ \ 1\ \ 1)t$

,

Table 4: The $d = 3$ regular triple palindrome family in bases $b, b + 1, b + 2$.

where

$$f(b) = f(8 + 2t) = 178, 373, 676, 1111, 1702, 2473, 3448, 4651, 6106, 7837, 9868, \ldots.$$

Recall the normal form families $n_1 = n_1(x)$ and $n_2 = n_2(x, y)$ from Theorem 2.14. Notice that the family $n_2(t + 2, t)$ is equivalent to the family from Table 4.

In other words, the intersection of the family $n_2(x, y)$ in bases $b, b + 1$ with the family $n_2(x, y)$ in bases $b + 1, b + 2$ is the family $n_2(t + 2, t)$ in bases $b, b + 1, b + 2$. Similarly, intersecting the family $n_1(x)$ in bases $b, b + 1$ with the family $n_1(x)$ in bases $b + 1, b + 2$ is just a single non-family example $n_1(2) = 300$ in bases $7, 8, 9$.

We can reduce double consecutive palindrome families ($k = 2$) to triple consecutive palindrome families ($k = 3$) for other cases of digits d in a similar fashion.

That is, $d = 3$ and $d = 5$ families from Theorem 2.18 follow from Theorem 2.14 and Theorem 2.16, respectively. The $d = 7$ families from Theorem 2.18 follow from the reduction of $d = 7$ solutions to systems from Corollary 2.9.

Note that the $d = 7$ digit case was also independently solved by Alekseyev (see [2]). The seven digit families from Theorem 2.18 can be rewritten as:

$(d = 7)$ $2-\text{periodic} f(b)$

$b = 74 + 2t$	$f(b) = \frac{1}{2}(6 + 25b + 55b^2 + 73b^3 + 55b^4 + 25b^5 + 7b^6 + b^7)$
$b + 0$	$(40\ 49\ 64\ 73\ 64\ 49\ 40) + (1\ 1\ 1\ 1\ 1\ 1\ 1)t$
$b + 1$	$(37\ 39\ 37\ 36\ 37\ 39\ 37) + (1\ 1\ 1\ 1\ 1\ 1\ 1)t$
$b + 2$	$(34\ 50\ 10\ 74\ 10\ 50\ 34) + (1\ 1\ 1\ 1\ 1\ 1\ 1)t$

$(d = 7)$ $6-\text{periodic} f(b)$'s

$b = 56 + 6t$	$f(b) = \frac{1}{6}(4 + 25b + 55b^2 + 73b^3 + 55b^4 + 25b^5 + 7b^6 + b^7)$
$b + 0$	$(10\ 32\ 18\ 40\ 18\ 32\ 10) + (1\ 3\ 1\ 3\ 1\ 3\ 1)t$
$b + 1$	$(9\ 29\ 9\ 28\ 9\ 29\ 9) + (1\ 3\ 1\ 3\ 1\ 3\ 1)t$
$b + 2$	$(8\ 33\ 0\ 41\ 0\ 33\ 8) + (1\ 3\ 1\ 3\ 1\ 3\ 1)t$
$b = 173 + 6t$	$f(b) = \frac{1}{6}(28 + 94b + 175b^2 + 217b^3 + 175b^4 + 91b^5 + 28b^6 + 4b^7)$
$b + 0$	$(120\ 15\ 58\ 36\ 58\ 15\ 120) + (4\ 0\ 1\ 0\ 1\ 0\ 4)t$
$b + 1$	$(116\ 1\ 29\ 1\ 29\ 1\ 116) + (4\ 0\ 1\ 0\ 1\ 0\ 4)t$
$b + 2$	$(112\ 15\ 0\ 36\ 0\ 15\ 112) + (4\ 0\ 1\ 0\ 1\ 0\ 4)t$
$b = 278 + 6t$	$f(b) = \frac{1}{6}(32 + 125b + 275b^2 + 365b^3 + 275b^4 + 125b^5 + 35b^6 + 5b^7)$
$b + 0$	$(237\ 159\ 277\ 199\ 277\ 159\ 237) + (5\ 3\ 5\ 3\ 5\ 3\ 5)t$
$b + 1$	$(232\ 142\ 232\ 137\ 232\ 142\ 232) + (5\ 3\ 5\ 3\ 5\ 3\ 5)t$
$b + 2$	$(227\ 160\ 187\ 200\ 187\ 160\ 227) + (5\ 3\ 5\ 3\ 5\ 3\ 5)t$

.

$(d = 7)$ $12-\text{periodic} f(b)$'s

$b = 37 + 12t$	$f(b) = \frac{1}{12}(10 + 68b + 193b^2 + 269b^3 + 187b^4 + 71b^5 + 16b^6 + 2b^7)$
$b + 0$	$(7\ 24\ 31\ 22\ 31\ 24\ 7) + (2\ 6\ 5\ 0\ 5\ 6\ 2)t$
$b + 1$	$(6\ 20\ 15\ 36\ 15\ 20\ 6) + (2\ 6\ 5\ 12\ 5\ 6\ 2)t$
$b + 2$	$(5\ 23\ 6\ 14\ 6\ 23\ 5) + (2\ 6\ 5\ 0\ 5\ 6\ 2)t$
$b = 117 + 12t$	$f(b) = \frac{1}{12}(66 + 256b + 543b^2 + 703b^3 + 537b^4 + 253b^5 + 72b^6 + 10b^7)$
$b + 0$	$(103\ 79\ 113\ 58\ 113\ 79\ 103) + (10\ 6\ 7\ 0\ 7\ 6\ 10)t$
$b + 1$	$(98\ 61\ 68\ 117\ 68\ 61\ 98) + (10\ 6\ 7\ 12\ 7\ 6\ 10)t$
$b + 2$	$(93\ 78\ 30\ 50\ 30\ 78\ 93) + (10\ 6\ 7\ 0\ 7\ 6\ 10)t$
$b = 289 + 12t$	$f(b) = \frac{1}{12}(10 + 80b + 283b^2 + 419b^3 + 277b^4 + 89b^5 + 16b^6 + 2b^7)$
$b + 0$	$(49\ 151\ 288\ 34\ 288\ 151\ 49) + (2\ 6\ 11\ 0\ 11\ 6\ 2)t$
$b + 1$	$(48\ 147\ 265\ 285\ 265\ 147\ 48) + (2\ 6\ 11\ 12\ 11\ 6\ 2)t$
$b + 2$	$(47\ 150\ 249\ 26\ 249\ 150\ 47) + (2\ 6\ 11\ 0\ 11\ 6\ 2)t$

Table 5: $d = 7$ regular triple palindrome families in bases $b, b + 1, b + 2$.

For example, the smallest term of the greatest $d = 7$ regular triple palindrome family is $n = f(289 + 12 \cdot 0) = 28854914566144178$.

The smallest (overall) $d = 7$ regular triple palindrome does not belong to any of the seven families given in Table 5. Instead, it is one of the twelve $d = 7$ non-family examples and equals 3360633.

2.3.2 Long digit cases

A regular triple palindrome is "long" if it is not "short". That is, it has $d \geq 9$ digits. Unexpectedly to us, it appears that long digit cases of regular triple palindromes do not exist. We would like to thank Alekseyev for verifying this for $d = 9, 11, 13$. We believe this is also in the case for all $d \geq 15$.

We introduce the idea of a period. We say that a regular triple palindrome in number bases $b, b+1, b+2$ belongs to a period p (infinite) family of normal forms if $b = b_0 + pt$ for some constants b_0, p and $t \in \mathbb{N}_0$. In Theorem 2.18 (on short palindromes), we can see that periods are either $2, 4, 6$ or 12. Additionally, the greatest b_0 among families from Theorem 2.18 is $b_0 = 289$. One would expect that neither the p nor the b_0 can be overly large.

It is possible to computationally set lower bounds on period $p \geq p_0$ and b_0. We observe p_0 consecutive number bases greater than b_0 and solve systems from Corollary 2.9 for all d digit double palindromes in those specific bases, one fixed base at a time.

We reduce the obtained double palindromes to triple palindromes. If there are no triple palindromes, then all (infinite) families either appear after b_0 or have periods greater than p_0 or do not exist.

For example, we obtained that for $d = 15, 17, 19$ the smallest (infinite) family either appears at $b_0 > 10^{12}$ or has $p_0 > 1000$ or does not exist. Its existence seems highly unlikely when we compare p and b_0 to periods and bases of short families from Theorem 2.18.

Maybe it is possible to establish theoretical upper bounds on p_0, b_0. This would allow us to computationally prove that a given (long) d digit case has at most finitely many regular triple palindromes, by improving the lower bounds. However, in this way one could resolve only finitely many cases of d.

To prove that $d \geq 9$ digit numbers (in the corresponding number bases) cannot be regular triple palindromes, remains an open problem. Our conjecture is:

Conjecture 2.19 (Long regular triple palindromes cannot exist). *If the number $n \in \mathbb{N}$ is a $d \geq 9$ digit regular double palindrome in number bases $b, b+1$, then it cannot be palindromic in number bases $b-1, b+2$.*

This was verified computationally for $d = 9, 11, 13$ thanks to Alekseyev. It remains to consider the case $d \geq 15$. If this conjecture is true, then we have found all regular triple palindromes as given in Theorem 2.18. It can be shown that none of examples from Theorem 2.18 can be palindromic in the fourth consecutive number base.

Therefore, the conjecture would imply that the consecutive palindromes in $k \geq 4$ consecutive number bases cannot be regular. That is, if they exist, they must be irregular. Recall the definition that unlike regular consecutive palindromes, irregular consecutive palindromes do not have the same number of digits d in all consecutive palindromic number bases.

2.4 Four or more number bases

In this chapter, we discuss the regular palindromes in four consecutive number bases, as well as the possibility of irregular examples. We would argue that they do not exist. Conjecture 2.19 in the previous chapter would imply that they cannot be regular, and Conjecture 3.1 in the next chapter would imply that they cannot be irregular. If both conjectures are proven, it would imply that a number cannot be palindromic in four or more consecutive number bases. To recap, here are the smallest $k = 1, 2, 3$ examples:

k	n	palindromic forms
1	3	$(1,1)_2$
2	10	$(1,0,1)_3 = (2,2)_4$
3	178	$(4,5,4)_6 = (3,4,3)_7 = (2,6,2)_8$
4	?	?

Table 6: The smallest $k = 1, 2, 3$ consecutive palindromes.

If there is a counterexample to either of those two conjectures, then the smallest $k = 4$ example would either be regular and have more than 13 digits in corresponding number bases or be irregular and greater than 10^{12}. Since we do not have any example for $k = 4$ palindromes, we tried computationally searching for "almost-($k = 4$) palindromes". That is, numbers that are palindromic in $b, b+3$ bases and in either the $b+1$ base or the $b+2$ base. The $d > 3$ "near examples" we could find are:

$$\begin{aligned}
1111 &= (1,1,1,1)_{10} & &= (7,8,7)_{12} & &= (6,7,6)_{13} \\
712410 &= (10,13,14,13,10)_{16} & &= (6,14,2,14,6)_{18} & &= (5,8,16,8,5)_{19} \ . \\
1241507 &= (14,14,11,14,14)_{17} & &= (11,14,15,14,11)_{18} & &= (7,15,3,15,7)_{20}
\end{aligned}$$

One can see only one irregular $d = (4, 3, 3)$ example, and two regular $d = 5$ examples. But we already knew that short ($d \leq 7$) consecutive palindromes cannot be palindromic in four consecutive number bases (write normal forms from Theorem 2.18 and Theorem 3.2 in the fourth consecutive base).

That is, not only we do not have any $k = 4$ examples, we also could not find any meaningful "near-($k = 4$) examples" either.

All results known so far motivate the conjecture:

Conjecture 2.20 (Quadruple and beyond consecutive palindromes)**.** *A positive integer cannot be palindromic in four or more consecutive number bases.*

To resolve this, it remains to prove Conjectures 2.19, 3.1. In the following chapter, we discuss irregular consecutive palindromes and state the second conjecture.

Alternatively, the problem can also be settled if one can find all "near examples": numbers palindromic in 3 out of 4 consecutive number bases.

3 Irregular consecutive palindromes

According to Definition 1.8, a consecutive palindrome is irregular if it does not have an equal number of digits in its consecutive number bases. We will say that it is a d digit example if it has d digits in the smallest number base b, out of the consecutive number bases $b, b+1, \ldots, b+k-1$. To find all d digit examples for some fixed digit case, we can solve systems similar to those presented in the chapter about regular double palindromes. The only difference is that we now have to consider multiple cases of digit tuples. There is nothing significantly new about this process compared to the regular palindromes, so we will just list our results below.

The irregular double palindromes are very rare compared to regular palindromes. Up to 10^{12}, we computationally confirmed that there are 74 irregular double palindromes. The terms are listed in the OEIS sequence A327810.

But, none of them extend to a third consecutive number base. In other words, we do not know if an irregular palindrome can be palindromic in three or more number bases. It could be that the smallest example is simply very large. Regardless, we state the problem of irregular palindromes as a conjecture:

Conjecture 3.1 (Irregular consecutive palindromes)**.** *If the number $n \in \mathbb{N}$ is a irregular double palindrome in number bases $b, b+1$, then it cannot be palindromic in number bases $b-1, b+2$.*

This means that, whether or not $n \in \mathbb{N}$ can be irregularly palindromic in three or more consecutive number bases remains an open problem.

We can confirm that for small cases of digits. In contrast to Theorem 2.18 giving all short regular palindromes, here we give all short irregular palindromes.

Theorem 3.2 (Short irregular palindromes)**.** *The number $n \in \mathbb{N}$ is a $d \leq 7$ digit irregular double palindrome in bases $b, b+1$ if and only if it belongs to one of the following* 11 *examples (normal forms):*

$$\begin{array}{lll}
10 & = (1,0,1)_3 & = (2,2)_4 \\
130 & = (1,1,2,1,1)_3 & = (2,0,0,2)_4 \\
651 & = (1,0,1,0,1)_5 & = (3,0,0,3)_6 \\
2997 & = (1,1,5,1,1)_7 & = (5,6,6,5)_8 \\
6886 & = (1,0,4,0,1)_9 & = (6,8,8,6)_{10} \\
9222 & = (2,1,0,0,0,1,2)_4 & = (2,4,3,3,4,2)_5 \\
26691 & = (1,3,2,3,2,3,1)_5 & = (3,2,3,3,2,3)_6 \\
27741 & = (1,3,4,1,4,3,1)_5 & = (3,3,2,2,3,3)_6 \\
626626 & = (1,1,5,4,5,1,1)_9 & = (6,2,6,6,2,6)_{10} \\
1798303 & = (1,0,1,9,1,0,1)_{11} & = (7,2,8,8,2,7)_{12} \\
1817179 & = (1,0,3,1,3,0,1)_{11} & = (7,3,7,7,3,7)_{12}
\end{array}.$$

Table 7: All short examples of irregular triple palindromes in bases $b, b+1$.

You can notice that there are only $1, 4, 6$ examples of $d = 3, 5, 7$ digit irregular consecutive palindromes, respectively. That is, in contrast to the regular consecutive palindromes which came in forms of infinite families, we have only finitely many irregular examples per digit case.

Note that Corollary 2.12 that tells us that there are infinitely many regular double palindromes for any given odd digit case d. Here, we can show that for any given odd digit case d there are only finitely many examples of irregular double palindromes.

Theorem 3.3 (see [1])**.** *For every fixed $d = 2l+1, l \in \mathbb{N}$ odd case of digits, there are at most finitely many d digit irregular double palindromes.*

The proof relies on the linearization approach given in Subsection 2.1.2. We again acknowledge and thank Alekseyev for suggesting the linearization approach. Here we additionally thank him for presenting the argument, which we will cite below.

Proof:
Let $d_1 < d_2$ be digits of the irregular double palindrome in number bases $b, b+1$. For an infinite set of examples to be possible for some digit case, values of b cannot be bounded. If $d_1 - d_2 > 1$, then the base b normal form represents asymptotically at least b times greater number than the base $b+1$ normal form. Therefore, we only need to consider the case when $d_1 - d_2 = 1$.

Due to Lemma 2.2 we can assume that d_1 is odd. Let $d_1 = d = 2l+1, d_2 = d-1 = 2l, l \in \mathbb{N}$. We have the normal forms

$$\begin{aligned} n &= \sum_{i=1}^{2l+1} a_i b^{2l+1-i} = \sum_{i=1}^{2l} c_i (b+1)^{2l-i} \\ &= \sum_{i=0}^{l-1} a_i (b^i + b^{2l-i}) + a_l b^l = \sum_{i=0}^{l-1} c_i ((b+1)^i + (b+1)^{2l-1-i}), \end{aligned}$$

where $a_0 \in [1, b-1]$, $c_0 \in [1, b]$, $a_i \in [0, b-1]$ and $c_i \in [0, b]$, for $i \in \{1, 2, \ldots, l\}$. Applying the same idea as in the linearization method from Subsection 2.1.2, we obtain a similar system to (2.8). We are only interested in the following part of it:

$$\begin{cases} a_0 = -k_d \\ a_1 = -\frac{d}{2}k_0 b + k_1 b - k_0 - \frac{d}{2}k_d \in [0, b-1] \\ c_0 = a_1 - k_d b + k_{d-1} \in [0, b] \end{cases},$$

where k_d, k_{d-1}, k_1 are integers whose bounds depend only on d.

"To keep $a_1 \in [0, b-1]$ and $c_0 \in [1, b]$ for large b, the coefficients of b in a_1 and c_0 must be between 0 and 1. Together with $a_0 \geq 1$ (i.e. $k_d \leq -1$) this implies that $k_d = -1$ and the coefficient of b in a_1 and c_0 equal 1 and 0, respectively. Then, however, a_1 is a half-integer, which is impossible. Thus, an infinite series of examples does not exist." − Alekseyev.

□

We haven't explored the asymptotics of these numbers. In other words, "What is the greatest d digit irregular consecutive palindrome?" remains an open problem for general d.

4 Conclusion

Here we will summarize the results given in this paper.

- If $n > 6$ is strictly non-palindromic, then it must be prime (Proposition 1.5).
- Consecutive palindromes must have an odd number of digits in their corresponding consecutive number bases (Theorem 2.3).
- Let $d > 1$ be a fixed odd number. There are at least infinitely many regular d digit consecutive palindromes and at most finitely many irregular d digit consecutive palindromes (Corollary 2.12 and Theorem 3.3).
- Searching for consecutive palindromes when the number of digits is fixed is equivalent to solving finitely many systems of linear Diophantine equations. We characterize all regular $d = 3, 5$ double palindromes (Theorems 2.14, 2.16) and all regular $d = 3, 5, 7$ triple palindromes (Theorem 2.18). We (re)compute all irregular palindromes up to 10^{12} and confirm the first 74 terms listed in the OEIS sequence A327810.

Additionally, we provide arguments and state the following conjectures:

- All irregular consecutive palindromes are double ($k = 2$) (Conjecture 3.1).
- The double consecutive palindromes are not "*nice enough*" to be given a closed form.
- The triple consecutive palindromes have all been found and are given in Theorem 2.18.
- The quadruple (or more) consecutive palindromes do not exist (Conjecture 2.20).

Problems we haven't explored, among other things, include the asymptotics of greatest irregular consecutive palindromes per digit case and finding infinite families of regular double palindromes that span across all digit cases (Problem 2.13).

Finally, we have learned a lot about consecutive palindromes. But, the main question behind this topic remains unsolved, i.e.,

"Can a positive integer be palindromic in more than three consecutive number bases?"

The question was originally asked in early 2017 (by the author of this paper), as far as we (the author) know(s) (see [16]). Of course, as stated in the definitions, trivial solutions like using one-digit numbers in corresponding number bases, are not being considered.

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